Energy and Momentum of the Metric Field

Overview

In General Relativity Theory, the gravitational potential of Newton's theory is replaced by the metric field of four-dimensional spacetime, and the gravitational force is replaced by the Christoffel symbols (which in essence are consisting of the metric field's derivatives with respect to the four spacetime coordinates). Accordingly the metric field can — like the gravitational potential in Newton's theory — store energy and momentum, and exchange it with other fields. In particular, the energy-stress-tensor of the metric field, and the "dynamic" energy-stress-tensors of the other fields, which are contained within spacetime, will be evaluated.

1. Notation

Starting point of our evaluation is Einstein's field equation

$$R_{\mu\nu} - \frac{R}{2}\, g_{\mu\nu} + \Lambda\, g_{\mu\nu} = -\frac{8\pi\, G}{c^4}\, T_{\mu\nu}\ , \tag{1}$$

in which $(R_{\mu\nu})$ is the Ricci-tensor, $R \equiv g^{\mu\nu} R_{\mu\nu}$ is the Ricci-scalar, $(g_{\mu\nu})$ is the metric tensor, Λ is the cosmological constant, G is the gravitation constant, and c is the speed of light. $(T_{\mu\nu})$ is the energy-density-momentumdensity-tensor of the fields, which are contained within spacetime (i. e. all fields with exception of the metric field itself). For this tensor, the shorter names energy-momentum-tensor or energy-tensor or energy-stress-tensor are conventional. We will simply call it ES-tensor in most cases.

With one exception, we use the same letters for tensors and for their contractions:

$$g^{\mu\nu} g^{\rho\sigma} R_{\rho\mu\sigma\nu} = g^{\mu\nu} R^{\sigma}{}_{\mu\sigma\nu} = g^{\mu\nu} R_{\mu\nu} = R^{\nu}{}_{\nu} = R \tag{2}$$

If we want to emphasize, that we are talking about the complete tensor but not just about one of it's components, we write $(R_{\mu\nu})$. But in a simplifying notation, we often will use $R_{\mu\nu}$ as indication of the complete tensor. Then it is visible only from the context, whether the tensor or just one of it's components is meant. In the case of vectors, we use the notation $A \equiv (A^{\nu})$.

The one exception is the metric tensor. For it's contraction

$$g^{\mu\nu} g_{\mu\nu} = g^{\nu}{}_{\nu} = 4 \tag{3}$$

we will *not* use the letter g. Instead

$$g \equiv \det(g_{\alpha\beta}) = \Big|(g_{\alpha\beta})\Big| \tag{4}$$

is defined to be the metric tensor's determinant. The metric tensor is symmetric, $g_{\mu\nu} = g_{\nu\mu}$. Therefore it can be transformed at any space-time-point P (but in general not globally) into a diagonal matrix. Furthermore the transformation can (at any point, but not globally) be chosen such, that the Minkowski-metric holds. We define it in the form

$$(\eta_{\alpha\beta}) \equiv \mathrm{diag}(+1, -1, -1, -1) \ . \tag{5}$$

The coordinate system with the metric $\eta_{\alpha\beta}$ is the local inertial system LS (i. e. the coordinate system of the tangent space at point P). The LS is a laboratory system of a free falling volume, which's extension in space and time is finite, because otherwise tidal forces would show up. Strictly speaking, the LS must even be infinitesimal small. The LS differs from the inertial systems of Special Relativity Theory by the absence of gravitational forces.

The derivative with respect to x^μ is marked either by a lower case d or by a stroke in front of the index:

$$\frac{\mathrm{d}\,A^\nu}{\mathrm{d}\,x^\mu} \equiv \mathrm{d}_\mu A^\nu \equiv A^\nu_{|\mu} \qquad\qquad \frac{\mathrm{d}\,A_\nu}{\mathrm{d}\,x^\mu} \equiv \mathrm{d}_\mu A_\nu \equiv A_{\nu|\mu} \ . \tag{6}$$

Ricci and Levi-Civita have shown, that the metric $g_{\mu\nu}$ can be related at any point of spacetime to the Minkowski-metric $\eta_{\alpha\beta}$ of the tangent space at this point by means of a "tetrad" (four-leg) [1]. The tetrad consists of four covariant unit vectors $\overrightarrow{e}_\mu$ with $\mu = 0, 1, 2, 3$, which span the tangent space. While $\mathrm{d}x^\mu$ is just one component of the four-dimensional vector $\mathrm{d}x$, each one of the four vectors $\overrightarrow{e}_\mu$ is a complete four-dimensional vector. These vectors have been written as an exception with arrows, to emphasize this fact. All other four-dimensional vectors, for example $\mathrm{d}x$, can be discerned only by the absence of a component index. The unit vectors, which are dual to the four unit vectors $\overrightarrow{e}_\mu$, are defined due to the relation

$$\vec{e}^{\,\nu}\,\vec{e}_{\mu}=\vec{e}^{\,\nu\alpha}\,\vec{e}_{\mu}^{\,\beta}\eta_{\alpha\beta}=g^{\nu}{}_{\mu}=\eta^{\nu}{}_{\mu} \tag{7a}$$

For them, the relations

$$\vec{e}^{\,\mu}{}_{\alpha}\,\vec{e}_{\mu}{}^{\beta}=\eta_{\alpha}{}^{\beta}=g_{\alpha}{}^{\beta} \tag{7b}$$

$$\vec{e}_{\nu}\,\vec{e}_{\mu}=e_{\nu}{}^{\alpha}\,e_{\mu}{}^{\beta}\eta_{\alpha\beta}=g_{\nu\mu} \tag{7c}$$

hold. In particular, for the Nabla-operator

$$\nabla=\vec{e}^{\,\mu}\,\mathrm{d}_{\mu} \tag{8}$$

holds. Thus the divergence of a contravariant vector field $A(x)$ is

$$\begin{aligned}\nabla A &= \vec{e}^{\,\mu}\,\mathrm{d}_{\mu}\vec{e}_{\nu}A^{\nu}=\vec{e}^{\,\mu}\,\vec{e}_{\nu}\mathrm{d}_{\mu}A^{\nu}+\vec{e}^{\,\mu}A^{\nu}\mathrm{d}_{\mu}\vec{e}_{\nu}\\ &= g^{\mu}{}_{\nu}\mathrm{d}_{\mu}A^{\nu}+A^{\alpha}g^{\mu}{}_{\nu}\underbrace{\vec{e}^{\,\nu}\mathrm{d}_{\mu}\vec{e}_{\alpha}}_{\Gamma^{\nu}_{\mu\alpha}}=g^{\mu}{}_{\nu}\Big(\underbrace{\mathrm{d}_{\mu}A^{\nu}+\Gamma^{\nu}_{\mu\alpha}A^{\alpha}}_{\mathrm{D}_{\mu}A^{\nu}}\Big)\ .\end{aligned} \tag{9}$$

In the last line, the covariant derivative

$$\mathrm{D}_{\mu}A^{\nu}\equiv\mathrm{d}_{\mu}A^{\nu}+\Gamma^{\nu}_{\mu\alpha}A^{\alpha} \tag{10}$$

has been defined, which — different from the "normal" derivative $\mathrm{d}_{\mu}A^{\nu}$ — is a tensor. The covariant derivative is marked either by the letter D or by two parallel strokes in front of the index:

$$\begin{aligned}\mathrm{D}_{\mu}A^{\nu}&\equiv A^{\nu}_{||\mu}\equiv\mathrm{d}_{\mu}A^{\nu}+\Gamma^{\nu}_{\mu\alpha}A^{\alpha}\\ \mathrm{D}_{\mu}A_{\nu}&\equiv A_{\nu||\mu}\equiv\mathrm{d}_{\mu}A_{\nu}-\Gamma^{\alpha}_{\mu\nu}A_{\alpha}\end{aligned} \tag{11}$$

The direct computation of the Christoffel-symbols results into [2, Kap. 11]

$$\Gamma^{\beta}_{\nu\alpha}\equiv\vec{e}^{\,\beta}\mathrm{d}_{\nu}\vec{e}_{\alpha}=\frac{g^{\beta\lambda}}{2}\Big(\frac{\mathrm{d}\,g_{\nu\lambda}}{\mathrm{d}\,x^{\alpha}}+\frac{\mathrm{d}\,g_{\alpha\lambda}}{\mathrm{d}\,x^{\nu}}-\frac{\mathrm{d}\,g_{\alpha\nu}}{\mathrm{d}\,x^{\lambda}}\Big)\ . \tag{12}$$

In the LS, the Christoffel symbols $\Gamma^{\nu}_{\mu\alpha}$ vanish. Therefore in the LS the covariant derivative and the "normal" derivative are identical.

The curvature tensor $R^{\mu}{}_{\rho\sigma\tau}$ is defined as the difference

$$\begin{aligned} R^{\mu}{}_{\rho\sigma\tau}A_{\mu} &\equiv A_{\rho||\tau||\sigma} - A_{\rho||\sigma||\tau} \\ R^{\mu}{}_{\rho\sigma\tau}A_{\mu} &\overset{[2,\,(18.8)]}{=} \frac{\mathrm{d}\Gamma^{\mu}_{\rho\sigma}}{\mathrm{d}x^{\tau}} - \frac{\mathrm{d}\Gamma^{\mu}_{\rho\tau}}{\mathrm{d}x^{\sigma}} + \Gamma^{\nu}_{\rho\sigma}\Gamma^{\mu}_{\nu\tau} - \Gamma^{\nu}_{\rho\tau}\Gamma^{\mu}_{\nu\sigma}\,. \end{aligned} \tag{13}$$

The Ricci-tensor, which is showing up in the field equation (1), is found due to contraction of the curvature tensor's indices μ and σ:

$$R_{\rho\tau} \equiv R^{\mu}{}_{\rho\mu\tau} = \frac{\mathrm{d}\Gamma^{\mu}_{\rho\mu}}{\mathrm{d}x^{\tau}} - \frac{\mathrm{d}\Gamma^{\mu}_{\rho\tau}}{\mathrm{d}x^{\mu}} + \Gamma^{\nu}_{\rho\mu}\Gamma^{\mu}_{\nu\tau} - \Gamma^{\nu}_{\rho\tau}\Gamma^{\mu}_{\nu\mu} \tag{14}$$

2. The Lagrangian of Empty Space-Time

We want to evaluate the metric field's energy and momentum by means of the Lagrange-formalism. For that purpose, we firstly must find a Lagrangian, from which the field equation (1) can be derived due to Hamilton's principle of least action. The following considerations are mainly based onto [3, Chap. 19].

We use the notation $\mathcal{L}_{EH}$ for the Lagrangian of the empty metric field ("classical vacuum"). The index $_{EH}$ is to signify Einstein-Hilbert. We write the Lagrangian as a product

$$L\sqrt{|g|} \equiv \mathcal{L}_{EH} \tag{15}$$

with a further function L, which — like $\mathcal{L}_{EH}$ — is still unknown. Both L and $\mathcal{L}_{EH}$ have the dimension energy/volume. The volume element

$$\mathrm{d}^4x'\sqrt{|g'|} = \mathrm{d}^4x\sqrt{|g|} \tag{16}$$

is invariant under arbitrary coordinate transformations, i. e. a scalar, see [4, (18)] or [2, (17.11)]. We demand that the action

$$S=\int\limits_{\omega}\frac{\mathrm{d}^4x}{c}\,\underbrace{\sqrt{|g|}\,L}_{\mathcal{L}_{EH}}\ , \tag{17}$$

in which ω denotes a simply connected closed range of four-dimensional spacetime, must be a scalar as well. Therefore also L — different from $\mathcal{L}_{EH}$ — must be a scalar. This is the first condition, which we will use as a guideline in the search for L.

A second condition for L is resulting from this consideration: Einstein's field-equation of the vacuum depends on the metric tensor $(g_{\mu\nu})$, and quadratically on it's first derivatives, and linearly on it's second derivatives. Therefore we demand, that L as well shall depend on $(g_{\mu\nu})$ and it's first and second derivatives in the same manner, but not on it's derivatives of higher order. It is proved in [5, chap. 6.2], that the Ricci-scalar $R\equiv R^{\mu}{}_{\mu}$ is the only scalar, which is depending in this manner on $(g_{\mu\nu})$.

A third condition for L is resulting from dimensional considerations: The dimension of L must be

$$[L]=\frac{\text{energy}}{\mathrm{m}^3}=\frac{1}{\mathrm{m}^2}\,\frac{\mathrm{kg\,s}^2}{\mathrm{m}^3}\,\frac{\mathrm{m}^4}{\mathrm{s}^4}=[R]\,\frac{[c^4]}{[G]} \tag{18}$$
$$[R]=\mathrm{m}^{-2}\quad,\quad [c]=\frac{\mathrm{m}}{\mathrm{s}}\quad,\quad [G]=\frac{\mathrm{m}^3}{\mathrm{kg\,s}^2}\ .$$

As the fourth and last condition for L we stipulate, that Einstein's field theory shall reduce to Newton's theory of gravitation in the weak field limit.

The sum

$$L\equiv\frac{c^4}{16\pi G}\,(R-2\Lambda) \tag{19}$$

with an arbitrary constant Λ (which like R must be of dimension m^{-2}) meets all four conditions for L. Thus one finds the action

$$S = \int\limits_\omega \frac{\mathrm{d}^4x}{c} \, \underbrace{\frac{\sqrt{|g|}\, c^4}{16\pi G} \left(g^{\mu\nu} R_{\mu\nu} - 2\Lambda\right)}_{\mathcal{L}_{EH}} \ . \tag{20}$$

According to Hamilton's principle, the variation of S must be zero. The variation must be performed with respect to the components $g^{\mu\nu}$ of the metric field, and with respect to it's first and second space-time-derivatives, which are contained within the Ricci-tensor:

$$\delta S = \delta S_1 + \delta S_2 + \delta S_3 = 0 \tag{21}$$

$$\begin{aligned} \delta S_1 &\equiv \frac{c^3}{16\pi G} \int\limits_\omega \mathrm{d}^4x \left(\delta\sqrt{|g|}\right) \left(g^{\mu\nu} R_{\mu\nu} - 2\Lambda\right) \\ \delta S_2 &\equiv \frac{c^3}{16\pi G} \int\limits_\omega \mathrm{d}^4x \sqrt{|g|} \left(\delta g^{\mu\nu}\right) R_{\mu\nu} \\ \delta S_3 &\equiv \frac{c^3}{16\pi G} \int\limits_\omega \mathrm{d}^4x \sqrt{|g|}\, g^{\mu\nu} \left(\delta R_{\mu\nu}\right) \end{aligned} \tag{22}$$

Note that $\delta \int_\omega \mathrm{d}^4x = 0$, because we demand that $\delta g^{\mu\nu}$, $\delta\, \mathrm{d}_\alpha g^{\mu\nu}$, and $\delta\, \mathrm{d}_\alpha \mathrm{d}_\beta g^{\mu\nu}$ shall be zero on the border (and outside) of the compact four-dimensional spacetime volume ω. Only in the interior of ω the metric and it's derivatives are varied.

For the computation of δS_1, we need

$$\delta\sqrt{|g|} = \delta\sqrt{-g} = \frac{-\delta(g)}{2\sqrt{-g}} \ . \tag{23}$$

We apply Jacobi's formula

$$\delta \det(A^{\mu\nu}) = \det(A^{\mu\nu})\, \mathrm{Tr}\{(A^{\mu\nu})^{-1}\, \delta(A^{\mu\nu})\} \ , \tag{24}$$

which is valid for arbitrary square matrices $(A^{\mu\nu})$ with non-vanishing determinant. In case of the metric tensor, using $(g_{\mu\nu})^{-1} = (g^{\mu\nu})$ one gets

$$\delta(g) = g\,\mathrm{Tr}\{(g^{\mu\nu})\,\delta(g_{\mu\nu})\} = g\,g^{\mu\nu}\,\delta g_{\nu\mu}\ . \tag{25}$$

From this follows

$$\delta\sqrt{|g|} \overset{(23)}{=} \frac{1}{2}\sqrt{-g}\,g^{\mu\nu}\,\delta g_{\nu\mu} = -\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\,\delta g^{\nu\mu}\ . \tag{26}$$

We prove the last equation:

$$\begin{aligned}
\delta g^{\mu}{}_{\nu} = 0 = \delta g^{\mu\rho} g_{\rho\nu} = (\delta g^{\mu\rho})g_{\rho\nu} + g^{\mu\rho}\delta g_{\rho\nu} \quad &\Big|\cdot g^{\nu\sigma}\\
(\delta g^{\mu\rho})g_{\rho}{}^{\sigma} = -g^{\nu\sigma}g^{\mu\rho}\delta g_{\rho\nu} \quad &\Big|\,\mu\leftrightarrow\rho\\
\delta g^{\rho\sigma} = -g^{\nu\sigma}g^{\rho\mu}\delta g_{\mu\nu} \quad &\Big|\cdot A_{\rho\sigma}\\
A_{\rho\sigma}\,\delta g^{\rho\sigma} = -A^{\mu\nu}\delta g_{\mu\nu}\ ,&
\end{aligned} \tag{27}$$

with $(A^{\mu\nu})$ being an arbitrary tensor.

δS_2 is left unchanged. To compute δS_3, we firstly consider the curvature tensor of fourth order $R^{\sigma}{}_{\mu\rho\nu} =$ (13). It is transformed into a coordinate systems, which at some arbitrarily fixed point P has the metric (5). At this point the Christoffel-symbols then are zero, and the variation of the curvature tensor simplifies to

$$\begin{aligned}
\delta R^{\sigma}{}_{\mu\rho\nu} &= \frac{\mathrm{d}(\delta\Gamma^{\sigma}_{\mu\rho})}{\mathrm{d}x^{\nu}} - \frac{\mathrm{d}(\delta\Gamma^{\sigma}_{\mu\nu})}{\mathrm{d}x^{\rho}}\\
&= (\delta\Gamma^{\sigma}_{\mu\rho})_{|\nu} - (\delta\Gamma^{\sigma}_{\mu\nu})_{|\rho} \quad \text{in the LS at point } P\ .
\end{aligned} \tag{28}$$

In the LS, the covariant derivative and the “normal” derivative are identical.

$$\delta R^{\sigma}{}_{\mu\rho\nu} = (\delta\Gamma^{\sigma}_{\mu\rho})_{||\nu} - (\delta\Gamma^{\sigma}_{\mu\nu})_{||\rho}\ . \tag{29}$$

As $\delta R^{\sigma}{}_{\mu\rho\nu} = R'^{\sigma}{}_{\mu\rho\nu} - R^{\sigma}{}_{\mu\rho\nu}$, being the difference of two tensors, again is a tensor, this equation — which is called Palatini-equation — does not only hold in the LS at point P, but at any arbitrary point in any arbitrary coordinate system. Now σ and ρ are contracted

$$\delta R_{\mu\nu} = (\delta\Gamma^{\sigma}_{\mu\sigma})_{||\nu} - (\delta\Gamma^{\sigma}_{\mu\nu})_{||\sigma} \ , \tag{30}$$

and the result is inserted into

$$\delta S_3 = \frac{c^3}{16\pi G} \int\limits_{\omega} \mathrm{d}^4x \, \sqrt{|g|} \, \underbrace{g^{\mu\nu}\Big((\delta\Gamma^{\sigma}_{\mu\sigma})_{||\nu} - (\delta\Gamma^{\sigma}_{\mu\nu})_{||\sigma}\Big)}_{(g^{\mu\nu}\delta\Gamma^{\sigma}_{\mu\sigma} - g^{\mu\tau}\delta\Gamma^{\nu}_{\mu\tau})_{||\nu}} \ . \tag{31}$$

The metric tensor may be pulled into the bracket, because it's covariant derivative is zero. Furthermore, contracted indices in the last term have been re-named. As the variation of Γ is zero on the surface $A(\omega)$ of the volume ω, δS_3 must be zero. This can be shown by means of the generalization of Gauß' theorem to curved Riemann-space:

$$\int\limits_{\omega} \mathrm{d}^4x \, \sqrt{|g|} \, V^{\mu}{}_{||\mu} = \int\limits_{A(\omega)} \mathrm{d}^3x \, \sqrt{|g|} \, n_{\mu} V^{\mu} \tag{32}$$

$A(\omega)$ is indicating the three-dimensional surface of the four-dimensional volume ω, n is the unit vector orthogonal to this surface, V is an arbitrary vector field, and g is the determinant of the metric tensor in the system of the coordinates x.

Thus one eventually gets Einstein's field-equations of the vacuum:

$$\delta S \overset{(21)}{=} \frac{c^3}{16\pi G} \int\limits_{\omega} \mathrm{d}^4x \, \sqrt{|g|} \, \Big(R_{\mu\nu} - \frac{1}{2} \, g_{\mu\nu} \, (R - 2\Lambda)\Big) \, \delta g^{\nu\mu} = 0 \tag{33}$$

$$\Longrightarrow \quad R_{\mu\nu} - \frac{R}{2} \, g_{\mu\nu} + \Lambda \, g_{\mu\nu} = 0$$

This equation is (apart from the ES-tensor $T_{\mu\nu}$ of the other fields contained within space-time) identical to the field-equation (1). This confirms, that $\mathcal{L}_{EH} = (20)$ is a correct Lagrangian of empty space-time.

3. Conservation of the Metric Fields's Energy and Momentum

When Einstein in 1916 published the first review of the young General Relativity Theory [4], he put much emphasis onto the proof of energy conservation in the metric field, which in this theory is replacing Newton's gravitational potential. In §15 of his treatise he considers the conservation of energy and momentum in the metric field alone, that is in the case of curved, but empty space-time. In §16 through §18, he then enlarges the evaluation to the case, that the curved space-time is containing an electromagnetic field and/or material fields. He demonstrates, that in this case energy and momentum is exchanged inbetween the metric field and the other fields contained in it, such that conservation laws only hold for the metric field and it's contents together, but not for the metric field or the other fields alone. We will in this section closely follow Einstein's delineation.

Firstly we set the cosmological constant $\Lambda = 0$, and re-formulate the field equation (1):

$$R_{\mu\nu} - \frac{R}{2}\, g_{\mu\nu} \overset{(1)}{=} -\frac{8\pi\, G}{c^4}\, T_{\mu\nu} \quad \Big| \cdot g^{\mu\nu}$$

$$R - \frac{R}{2}\, \underbrace{g^{\mu\nu} g_{\mu\nu}}_{4} = -R = -\frac{8\pi\, G}{c^4}\, T$$

$$\Longrightarrow \qquad R_{\mu\nu} = -\frac{8\pi\, G}{c^4}\Big(T_{\mu\nu} - \frac{T}{2}\, g_{\mu\nu}\Big) \tag{34}$$

∗ In his equation (29a), Einstein notes the following important relation for the Christoffel-symbol with one contraction:

$$\Gamma^{\mu}_{\rho\mu} = \frac{g^{\mu\alpha}}{2}\frac{\mathrm{d}g_{\mu\alpha}}{\mathrm{d}x^{\rho}} = \frac{1}{2}\frac{\mathrm{d}\ln|g|}{\mathrm{d}x^{\rho}} \qquad \text{with } g \equiv \det(g_{\alpha\beta}) \tag{35}$$

∗ Using (35), the Ricci-tensor (14) can be written in the form

$$R_{\rho\tau} = \frac{1}{2}\frac{\mathrm{d}^2\ln|g|}{\mathrm{d}x^{\rho}\mathrm{d}x^{\tau}} - \frac{\mathrm{d}\Gamma^{\mu}_{\rho\tau}}{\mathrm{d}x^{\mu}} + \Gamma^{\nu}_{\rho\mu}\Gamma^{\mu}_{\nu\tau} - \Gamma^{\nu}_{\rho\tau}\frac{1}{2}\frac{\mathrm{d}\ln|g|}{\mathrm{d}x^{\nu}} \; . \tag{36}$$

∗ Einstein restricts his evaluation to coordinate systems, for which $|\det(g_{\mu\nu})| = |g| = 1$ holds. It becomes obvious from (35) and (36), that the complexity of the formulae is considerably reduced in this case. On page 815 Einstein assures to have conducted the evaluations for the case $|g| \neq 1$ as well, and to have achieved in principle identical results as in the case $|g| = 1$. “But I think, that the communication of my quite lengthly considerations on this topic would not be worthwhile, because there is nothing essentially new in those results.” (my translation) Note, that the condition $|g| = 1$ does not at all indicate a return to Minkowski-metric. While $|g| = 1$ holds for Minkowski-metric as well, Einstein is considering curved space-times with

$$\frac{\mathrm{d}g_{\mu\nu}}{\mathrm{d}x^{\sigma}} \neq 0 \quad , \quad \Gamma^{\mu}_{\nu\tau} \neq 0 \quad , \quad |\det(g_{\mu\nu})| = |g| = 1 \quad .$$

While the Christoffel-symbol in general is different from zero, it’s contracted form is zero in case $|g| = 1$, as is visible in (35).

∗ The field equations of space volumes, in which no type of energy except for gravitational energy is contained (“vacuum”), simplify to

$$R_{\mu\nu} \overset{(34),(36)}{=} -\frac{\mathrm{d}\Gamma^{\alpha}_{\mu\nu}}{\mathrm{d}x^{\alpha}} + \Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\nu\beta} = 0 \quad \text{if } |g| = 1 \; , \tag{37}$$

and the Lagrangian

$$\mathcal{L}_{EH} \overset{(20)}{=} \frac{\sqrt{|g|}}{2\kappa}\,(R - 2\Lambda) \qquad , \qquad \kappa \equiv \frac{8\pi G}{c^4}\ , \tag{38}$$

which has been checked in the previous section, becomes in this case with $\Lambda = 0$

$$\mathcal{L}_{EH} \overset{(36)}{=} \frac{1}{2\kappa}\Big(- g^{\mu\nu}\frac{\mathrm{d}\Gamma^{\beta}_{\mu\nu}}{\mathrm{d}x^{\beta}} + g^{\mu\nu}\Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\alpha\nu}\Big) \quad \text{if } |g| = 1\ . \tag{39}$$

As is well-known, the action integral's variation is unchanged, if the four-divergence of an arbitrary function of space-time is added to the Lagrangian, see e. g. [6, Chap. 3]. As the covariant derivative of the metric tensor is zero,

$$\mathrm{D}_{\beta} g^{\mu\nu}\Gamma^{\beta}_{\mu\nu} = g^{\mu\nu}(\mathrm{d}_{\beta}\Gamma^{\beta}_{\mu\nu} + \Gamma^{\beta}_{\beta\alpha}\Gamma^{\alpha}_{\mu\nu}) \overset{(35)}{=} g^{\mu\nu}\mathrm{d}_{\beta}\Gamma^{\beta}_{\mu\nu} \quad \text{if } |g| = 1$$

holds. Multiplying this four-divergence by $1/(2\kappa)$, and adding the result to the Lagrangian (39), one gets the Lagrangian

$$\mathcal{L} \equiv \frac{1}{2\kappa}\, g^{\mu\nu}\Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha} \quad \text{with } \kappa \equiv \frac{8\pi G}{c^4} \quad \text{if } |g| = 1\ . \tag{40}$$

This Lagrangian is Einstein's starting point. Firstly he checks explicitly, that the field equation (37) can be derived due to the variation of the action

$$\delta S = \delta \int\limits_{\omega} \frac{\mathrm{d}^4 x}{c}\sqrt{|g|}\,\frac{\mathcal{L}}{\sqrt{|g|}} = 0 \quad \text{if } |g| = 1\ .$$

ω is a simply connected, compact range of the four-dimensional space-time-continuum. Because on the boundary (and beyond) of

ω there is no variation, and because we restrict to $|g| = 1$, the Lagrangian $\mathcal{L}$ is the only factor in the action, which is varied.

$$\begin{aligned} 2\kappa\,\delta\mathcal{L} &= \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha}\,\delta g^{\mu\nu} + 2g^{\mu\nu}\Gamma^{\alpha}_{\mu\beta}\,\delta\Gamma^{\beta}_{\nu\alpha} \\ &= 2\Gamma^{\alpha}_{\mu\beta}\underbrace{\delta\left(g^{\mu\nu}\Gamma^{\beta}_{\nu\alpha}\right)}_{\frac{1}{2}\,\delta\left[g^{\mu\nu}g^{\beta\lambda}\left(\frac{\mathrm{d}\,g_{\nu\lambda}}{\mathrm{d}\,x^{\alpha}}+\frac{\mathrm{d}\,g_{\alpha\lambda}}{\mathrm{d}\,x^{\nu}}-\frac{\mathrm{d}\,g_{\alpha\nu}}{\mathrm{d}\,x^{\lambda}}\right)\right]} - \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha}\,\delta g^{\mu\nu} \end{aligned} \tag{41}$$

Comparing this with Einstein's equation at the bottom of page 804, one should note that we have defined the Christoffel-symbol (12) with opposite sign. As the Christoffel-symbol is symmetric in the both lower indices, in the underbraced expression μ and β, and consequently ν and λ may be permuted. For that reason, the both last terms of the underbraced expression vanish:

$$2\kappa\,\delta\mathcal{L} = \Gamma^{\alpha}_{\mu\beta}\,\delta\left[g^{\mu\nu}g^{\beta\lambda}\frac{\mathrm{d}\,g_{\nu\lambda}}{\mathrm{d}\,x^{\alpha}}\right] - \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha}\,\delta g^{\mu\nu} \tag{42}$$

Using

$$g^{\nu\tau}g^{\mu\rho}\,\frac{\mathrm{d}g_{\tau\rho}}{\mathrm{d}x^{\sigma}} = g^{\nu\tau}\,\underbrace{\frac{\mathrm{d}(g^{\mu\rho}g_{\tau\rho})}{\mathrm{d}x^{\sigma}}}_{0} - g^{\nu\tau}\,\frac{\mathrm{d}g^{\mu\rho}}{\mathrm{d}x^{\sigma}}\,g_{\tau\rho} = -\frac{\mathrm{d}g^{\mu\nu}}{\mathrm{d}x^{\sigma}} \tag{43a}$$

$$g_{\nu\tau}g_{\mu\rho}\,\frac{\mathrm{d}g^{\tau\rho}}{\mathrm{d}x^{\sigma}} = g_{\nu\tau}\,\underbrace{\frac{\mathrm{d}(g_{\mu\rho}g^{\tau\rho})}{\mathrm{d}x^{\sigma}}}_{0} - g_{\nu\tau}\,\frac{\mathrm{d}g_{\mu\rho}}{\mathrm{d}x^{\sigma}}\,g^{\tau\rho} = -\frac{\mathrm{d}g_{\mu\nu}}{\mathrm{d}x^{\sigma}}\ , \tag{43b}$$

one gets

$$2\kappa\,\delta\mathcal{L} = -\Gamma^{\alpha}_{\mu\nu}\,\delta\left(\frac{\mathrm{d}\,g^{\mu\nu}}{\mathrm{d}\,x^{\alpha}}\right) - \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha}\,\delta g^{\mu\nu}\ . \tag{44}$$

The differential quotients, which are needed for the computation of the canonical field equation, can be read-off from this equation:

$$\begin{aligned} \mathrm{d}_{\alpha}\,\frac{\partial\mathcal{L}}{\partial(\mathrm{d}_{\alpha}g^{\mu\nu})} - \frac{\partial\mathcal{L}}{\partial g^{\mu\nu}} &= 0 \\ -\mathrm{d}_{\alpha}\,\Gamma^{\alpha}_{\mu\nu} + \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha} &= 0 \quad \text{if } |g| = 1 \end{aligned} \tag{45}$$

This field equation is identical to (37). Thus $\mathcal{L} = (40)$ is indeed a correct Lagrangian under the condition $|g| = 1$. Now Einstein multiplies the field equation by $\mathrm{d}_\sigma g^{\mu\nu}$:

$$\begin{aligned}
0 &= (\mathrm{d}_\sigma g^{\mu\nu})\mathrm{d}_\alpha \frac{\partial\mathcal{L}}{\partial(\mathrm{d}_\alpha g^{\mu\nu})} - (\mathrm{d}_\sigma g^{\mu\nu})\frac{\partial\mathcal{L}}{\partial g^{\mu\nu}} \\
&= \mathrm{d}_\alpha(\mathrm{d}_\sigma g^{\mu\nu})\frac{\partial\mathcal{L}}{\partial(\mathrm{d}_\alpha g^{\mu\nu})} \underbrace{- \frac{\partial\mathcal{L}}{\partial(\mathrm{d}_\alpha g^{\mu\nu})}\mathrm{d}_\sigma\mathrm{d}_\alpha g^{\mu\nu} - \frac{\partial\mathcal{L}}{\partial g^{\mu\nu}}\mathrm{d}_\sigma g^{\mu\nu}}_{-\mathrm{d}_\sigma\mathcal{L}}
\end{aligned} \tag{46}$$

Here $\mathrm{d}_\alpha\mathrm{d}_\sigma g^{\mu\nu} = \mathrm{d}_\sigma\mathrm{d}_\alpha g^{\mu\nu}$ has been used. This is the continuity equation of the energy tensor ${t_\sigma}^\alpha$:

$$\mathrm{d}_\alpha {t_\sigma}^\alpha = 0 \tag{47}$$

$$\begin{aligned}
-2\kappa {t_\sigma}^\alpha &\equiv (\mathrm{d}_\sigma g^{\mu\nu})\frac{\partial\mathcal{L}}{\partial(\mathrm{d}_\alpha g^{\mu\nu})} - {g_\sigma}^\alpha\mathcal{L} \\
&\overset{(40)}{=} -(\mathrm{d}_\sigma g^{\mu\nu})\Gamma^\alpha_{\mu\nu} - {g_\sigma}^\alpha g^{\mu\nu}\Gamma^\tau_{\mu\beta}\Gamma^\beta_{\nu\tau} \qquad \text{if } |g| = 1
\end{aligned} \tag{48}$$

Note firstly, that the energy tensor's definition has the same form as the definitions of the energy tensors of other fields in canonical field theory in Minkowski-metric [6, Chap. 4]. But here we are considering metrics with $\mathrm{d}g^{\rho\sigma}/\mathrm{d}x^\tau \neq 0$, because otherwise ${t_\sigma}^\alpha = 0$ would hold. The factor -2κ has been inserted, to get the energy of Newton's gravitational field in the limit of weak gravitation. (Einstein considers that limit in §21 of his treatise.)

Note secondly, that $({t_\sigma}^\alpha)$ strictly speaking is not a tensor, because it is not form-invariant under arbitrary coordinate transformations: Remember the restriction $|g| = 1$. This restriction was visible already in the definition (40) of the Lagrangian. $\mathcal{L}/\sqrt{|g|}$ is not a Riemann-scalar under arbitrary transformations, but only under the additional condition $|g| = 1$. Still we will continue to call — though somewhat imprecisely — the matrix $({t_\sigma}^\alpha) = (48)$

energydensity-stress-tensor, or simply ES-tensor, of the metric field under the additional condition $|g|=1$.

Now Einstein wants to clarify, how the ES-tensor is related to the equation of the metric field. He firstly performs a short auxiliary computation:

$$\begin{aligned} g^{\nu\tau}\Gamma^{\mu}_{\sigma\tau} + g^{\mu\tau}\Gamma^{\nu}_{\sigma\tau} &= g^{\nu\tau}\frac{g^{\mu\rho}}{2}\Big(\frac{\mathrm{d}g_{\tau\rho}}{\mathrm{d}x^{\sigma}} + \frac{\mathrm{d}g_{\sigma\rho}}{\mathrm{d}x^{\tau}} - \frac{\mathrm{d}g_{\sigma\tau}}{\mathrm{d}x^{\rho}}\Big) + \\ &\quad + g^{\mu\tau}\frac{g^{\nu\rho}}{2}\Big(\frac{\mathrm{d}g_{\tau\rho}}{\mathrm{d}x^{\sigma}} + \frac{\mathrm{d}g_{\sigma\rho}}{\mathrm{d}x^{\tau}} - \frac{\mathrm{d}g_{\sigma\tau}}{\mathrm{d}x^{\rho}}\Big) \end{aligned} \tag{49}$$

Four terms on the right side of this equation compensate, because they differ only in the names of the contracted indices ρ and τ. Therefore

$$g^{\nu\tau}\Gamma^{\mu}_{\sigma\tau} + g^{\mu\tau}\Gamma^{\nu}_{\sigma\tau} = g^{\nu\tau}g^{\mu\rho}\frac{\mathrm{d}g_{\tau\rho}}{\mathrm{d}x^{\sigma}} \overset{(43)}{=} -\frac{\mathrm{d}g^{\mu\nu}}{\mathrm{d}x^{\sigma}} \tag{50}$$

holds. Using this result, the ES-tensor (48) can be written in the form

$$\begin{aligned} -2\kappa t_{\sigma}{}^{\alpha} &= (g^{\nu\tau}\Gamma^{\mu}_{\sigma\tau} + g^{\mu\tau}\Gamma^{\nu}_{\sigma\tau})\Gamma^{\alpha}_{\mu\nu} - g_{\sigma}{}^{\alpha}g^{\mu\nu}\Gamma^{\tau}_{\mu\beta}\Gamma^{\beta}_{\nu\tau} \\ \kappa t_{\sigma}{}^{\alpha} &= \tfrac{1}{2}g_{\sigma}{}^{\alpha}g^{\mu\nu}\Gamma^{\beta}_{\nu\tau}\Gamma^{\tau}_{\mu\beta} - g^{\mu\nu}\Gamma^{\beta}_{\nu\sigma}\Gamma^{\alpha}_{\mu\beta} \quad \text{if } |g|=1 \qquad (51) \\ \kappa t = \kappa t_{\sigma}{}^{\sigma} &= g^{\mu\nu}\Gamma^{\beta}_{\nu\tau}\Gamma^{\tau}_{\mu\beta} \quad \text{because of } g_{\sigma}{}^{\sigma}=4 \\ \kappa(t_{\sigma}{}^{\alpha} - \tfrac{1}{2}g_{\sigma}{}^{\alpha}t) &= -g^{\mu\nu}\Gamma^{\beta}_{\nu\sigma}\Gamma^{\alpha}_{\mu\beta} \quad \text{if } |g|=1\ . \qquad (52) \end{aligned}$$

Multiplication of the field equation (45) by $g^{\nu\sigma}$ results into

$$-g^{\nu\sigma}\frac{\mathrm{d}\Gamma^{\alpha}_{\nu\mu}}{\mathrm{d}x^{\alpha}} + g^{\nu\sigma}\Gamma^{\beta}_{\nu\alpha}\Gamma^{\alpha}_{\beta\mu} = 0 \quad \text{if } |g|=1\ . \tag{53}$$

The first term is

$$\begin{aligned} -g^{\nu\sigma}\frac{\mathrm{d}\Gamma^{\alpha}_{\nu\mu}}{\mathrm{d}x^{\alpha}} &= -\frac{\mathrm{d}(g^{\nu\sigma}\Gamma^{\alpha}_{\nu\mu})}{\mathrm{d}x^{\alpha}} + \frac{\mathrm{d}g^{\nu\sigma}}{\mathrm{d}x^{\alpha}}\Gamma^{\alpha}_{\nu\mu} \\ &\overset{(50)}{=} -\frac{\mathrm{d}(g^{\nu\sigma}\Gamma^{\alpha}_{\nu\mu})}{\mathrm{d}x^{\alpha}} - g^{\sigma\tau}\Gamma^{\nu}_{\alpha\tau}\Gamma^{\alpha}_{\nu\mu} - g^{\nu\tau}\Gamma^{\sigma}_{\alpha\tau}\Gamma^{\alpha}_{\nu\mu} \end{aligned} \tag{54}$$

The second term of this expression differs from the second term of the field equation only by the names of two contracted indices. Thus one gets the field equation

$$-\frac{\mathrm{d}\,(g^{\nu\sigma}\Gamma^{\alpha}_{\nu\mu})}{\mathrm{d}\,x^{\alpha}}\underbrace{-g^{\nu\tau}\Gamma^{\sigma}_{\alpha\tau}\,\Gamma^{\alpha}_{\nu\mu}}_{+\,\kappa\,(t_{\mu}{}^{\sigma}-\frac{1}{2}g_{\mu}{}^{\sigma}t)}=0 \quad \text{if } |g|=1\,. \tag{55}$$

From this equation it becomes immediately obvious, how the field equation shall be modified, if there are in addition to the metric field further fields, like e. g. an electromagnetic field or a material field: The ES-tensors $T_{\mu}{}^{\sigma}$ of the other fields must be added to the ES-tensor $t_{\mu}{}^{\sigma}$ of the metric field:

$$-\frac{\mathrm{d}\,(g^{\nu\sigma}\Gamma^{\alpha}_{\nu\mu})}{\mathrm{d}\,x^{\alpha}}=-\kappa\left(t_{\mu}{}^{\sigma}+T_{\mu}{}^{\sigma}-\tfrac{1}{2}g_{\mu}{}^{\sigma}(t+T)\right) \text{ if } |g|=1\,. \tag{56}$$

We shift all terms with t, but not the terms with T, to the equation's left side, and transform the equation back again from the form (55) to the form (53). Eventually multiplying both sides of the equation by $g_{\nu\sigma}$, one gets

$$R_{\mu\nu}=-\frac{\mathrm{d}\,\Gamma^{\alpha}_{\nu\mu}}{\mathrm{d}\,x^{\alpha}}+\Gamma^{\beta}_{\nu\alpha}\Gamma^{\alpha}_{\beta\mu}=-\kappa\,(T_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}T) \text{ if } |g|=1\,. \tag{57}$$

To evaluate the conservation of energy and momentum for the combination of the metric field and the material fields, Einstein contracts (56) with regard to the indices μ and σ, multiplies the result by $\frac{1}{2}g_{\mu}{}^{\sigma}$, and subtracts the result from the original equation (56):

$$-\frac{\mathrm{d}\,(g^{\nu\sigma}\Gamma^{\alpha}_{\nu\mu})}{\mathrm{d}\,x^{\alpha}}+\tfrac{1}{2}g_{\mu}{}^{\sigma}\frac{\mathrm{d}\,(g^{\nu\rho}\Gamma^{\alpha}_{\nu\rho})}{\mathrm{d}\,x^{\alpha}}=\kappa\,(t_{\mu}{}^{\sigma}+T_{\mu}{}^{\sigma}) \tag{58}$$

Then he computes the divergence of this equation with regard to the index σ. For the first term one finds

$$-\frac{\mathrm{d}^2(g^{\nu\sigma}\Gamma^{\alpha}_{\nu\mu})}{\mathrm{d}x^\sigma \mathrm{d}x^\alpha} = -\frac{\mathrm{d}^2}{\mathrm{d}x^\sigma \mathrm{d}x^\alpha}\Big[g^{\nu\sigma}\frac{g^{\alpha\tau}}{2}\Big(\frac{\mathrm{d}g_{\mu\tau}}{\mathrm{d}x^\nu}+\frac{\mathrm{d}g_{\nu\tau}}{\mathrm{d}x^\mu}-\frac{\mathrm{d}g_{\nu\mu}}{\mathrm{d}x^\tau}\Big)\Big]\ .$$

As this expression is invariant, if at the same time α is permuted with σ, and ν is permuted with τ, the first and the third term in the large round bracket compensate.

$$-\frac{\mathrm{d}^2(g^{\nu\sigma}\Gamma^{\alpha}_{\nu\mu})}{\mathrm{d}x^\sigma \mathrm{d}x^\alpha} = -\frac{\mathrm{d}^2}{\mathrm{d}x^\sigma \mathrm{d}x^\alpha}\Big[g^{\nu\sigma}\frac{g^{\alpha\tau}}{2}\frac{\mathrm{d}g_{\nu\tau}}{\mathrm{d}x^\mu}\Big] \overset{(43)}{=} \frac{1}{2}\frac{\mathrm{d}^3 g^{\sigma\alpha}}{\mathrm{d}x^\sigma \mathrm{d}x^\alpha \mathrm{d}x^\mu} \tag{59a}$$

The divergence of the second term in (58) with regard to the index σ is

$$\frac{1}{2}\frac{\mathrm{d}^2(g^{\nu\rho}\Gamma^{\alpha}_{\nu\rho})}{\mathrm{d}x^\mu \mathrm{d}x^\alpha} = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^\mu \mathrm{d}x^\alpha}\Big[g^{\nu\rho}\frac{g^{\alpha\tau}}{2}\Big(\frac{\mathrm{d}g_{\rho\tau}}{\mathrm{d}x^\nu}+\frac{\mathrm{d}g_{\nu\tau}}{\mathrm{d}x^\rho}-\frac{\mathrm{d}g_{\nu\rho}}{\mathrm{d}x^\tau}\Big)\Big]\ .$$

The last term is zero for any metric with $|g| = 1$ because of (35). The remaining expression is symmetric in ν and ρ. Therefore it can be combined to one term. Furthermore the contracted index ρ is renamed to σ:

$$\frac{1}{2}\frac{\mathrm{d}^2(g^{\nu\rho}\Gamma^{\alpha}_{\nu\rho})}{\mathrm{d}x^\mu \mathrm{d}x^\alpha} = \frac{1}{4}\frac{\mathrm{d}^2}{\mathrm{d}x^\mu \mathrm{d}x^\alpha} g^{\nu\sigma}g^{\alpha\tau}\cdot 2\,\frac{\mathrm{d}g_{\nu\tau}}{\mathrm{d}x^\sigma} \overset{(43)}{=} -\frac{1}{2}\frac{\mathrm{d}^3 g^{\sigma\alpha}}{\mathrm{d}x^\mu \mathrm{d}x^\alpha \mathrm{d}x^\sigma} \tag{59b}$$

Thus the divergence of the left side of (58) in total is zero. Consequently, the same must hold for the right side:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x^\sigma}\left({t_\mu}^\sigma + {T_\mu}^\sigma\right) = 0 \quad \text{if } |g| = 1} \tag{60}$$

This result proves: Conservation laws do hold neither for energy and momentum of the metric field alone, nor for energy and momentum alone of those fields which are contained within space-time. Instead only the sums of the energies and the sums of the momenta of the metric field and of it's contents are conserved, i. e. energy and momentum can be exchanged in-between the metric field and the fields contained in it.

We would like to get rid of the restricting condition $|g| = 1$, and prove instead

$$\frac{\mathrm{d}}{\mathrm{d}\,x^\sigma}\,(t_\mu{}^\sigma + T_\mu{}^\sigma) \overset{?}{=} 0 \qquad \text{with arbitrary } |g| \ .$$

But under closer examination [7] it turns out, that it is impossible to specify an unambiguous closed expression for the metric field's energy-stress-tensor.

To get at least a superficial impression of conservation of energy and momentum in the general case with arbitrary $|g|$, we first convert the field equation (57) into another form, which as well is often used — for example in (1): One performs the contraction

$$g^{\mu\nu}R_{\mu\nu} = R = -\kappa\,(\underbrace{g^{\mu\nu}T_{\mu\nu}}_{T} - \tfrac{1}{2}\,\underbrace{g^{\mu\nu}g_{\mu\nu}}_{4}\,T) = \kappa T \ ,$$

shifts the Ricci-scalar R to the left side, and thus gets the field equation in the form

$$R_{\mu\nu} - \frac{R}{2}\,g_{\mu\nu} = -\kappa\,T_{\mu\nu} = -\frac{8\pi G}{c^4}\,T_{\mu\nu} \ . \tag{61}$$

Consequently in any case

$$\mathrm{d}_\sigma\Big(\frac{R^{\sigma\tau}}{\kappa} - \frac{R}{2\kappa}\,g^{\sigma\tau}\Big) = -\mathrm{d}_\sigma T^{\sigma\tau} \ . \tag{62}$$

holds trivially. The tensor $R^{\sigma\tau}/\kappa - Rg^{\sigma\tau}/(2\kappa) \neq \tau^{\sigma\tau}$ is *not* the metric field's ES-tensor, but it's divergence obviously is identical to the divergence of the hypothetical, not explicitly formulable ES-tensor of the metric field.

In special relativity, the equations of continuity

$$T_{\mu\nu|\mu} = 0 \quad \text{with } \nu = 0, 1, 2, 3 \ \text{ if } g_{\mu\nu}(x) = \eta_{\mu\nu} \, \forall x \tag{63}$$

hold for the sum of all fields (including the gravitational field) contained within space-time. This is interpreted as the conservation of energy density and momentum density of the fields represented by $(T_{\mu\nu})$. In case of GRT, the energy and momenta which are stored in the metric field are not booked in $(T_{\mu\nu})$, but somewhere within the Ricci-tensor and the Ricci-scalar on the left side of equation (62).

In the limit of vanishing gravity (free falling laboratory), there exists at least in an infinitesimal neighborhood of any point in space-time a local coordinate system LS, for which Minkowski-metric is valid. Therefore in the LS, the curvature-tensor and consequently the tensor $R^{\sigma\tau}/\kappa - Rg^{\sigma\tau}/(2\kappa)$ are zero. Consequently

$$\mathrm{d}_\nu T^{\mu\nu} = 0 \quad \text{holds in the LS.} \tag{64}$$

The covariant tensor-equation, which reduces to this limit, and stays form-invariant under transformations into arbitrary accelerated reference systems, is

$$\mathrm{D}_\nu \, T^{\mu\nu} = \mathrm{d}_\nu \, T^{\mu\nu} + \Gamma^\mu_{\nu\alpha} T^{\alpha\nu} + \Gamma^\nu_{\nu\alpha} T^{\mu\alpha} = 0 \; . \tag{65}$$

Consequently one finds, using (62):

$$\boxed{\begin{aligned} \mathrm{d}_\nu \, T^{\mu\nu} &= -\Gamma^\mu_{\nu\alpha} T^{\alpha\nu} - \Gamma^\nu_{\nu\alpha} T^{\mu\alpha} \\ \mathrm{d}_\nu \left(\frac{R^{\mu\nu}}{\kappa} - \frac{R \, g^{\mu\nu}}{2\kappa} \right) &= +\Gamma^\mu_{\nu\alpha} T^{\alpha\nu} + \Gamma^\nu_{\nu\alpha} T^{\mu\alpha} \end{aligned}} \tag{66}$$

On the right sides of these both equations, the amount of energy- and momentum-density is listed, which is exchanged inbetween the metric field and it's contents. Conservation laws neither hold for the metric field alone nor for the material fields alone, but only for their combination. I. e. energy and momentum are exchanged in-between space-time and it's material contents. Only in the vacuum $T^{\mu\nu}=0$ the second equation simplifies to

$$\mathrm{d}_\nu\Big(\frac{R^{\mu\nu}}{\kappa}-\frac{R\,g^{\mu\nu}}{2\kappa}\Big)=0\quad\text{if } T^{\mu\nu}=0\ . \tag{67}$$

4. The Dynamic ES-Tensor

We have derived the equation of the free metric field from a Lagrangian, but then we have inserted the ES-tensor T "by hand" into equation (56). We now will try to derive the ES-tensor as well systematically from a Lagrangian.

A suitable Lagrangian has been proposed by Einstein in [8]. There he assumed, that the Lagrangian does depend on the metric field $g^{\tau\mu}$, on it's first and second derivatives $\mathrm{d}_\alpha g^{\tau\mu}$ and $\mathrm{d}_\alpha\mathrm{d}_\beta g^{\tau\mu}$, on the fields ϕ_r which are contained within spacetime, and on their first derivatives $\mathrm{d}_\alpha\phi_r$. (Two lines above his equation (1), there is an obvious typo.) He assumes in addition, that the Lagrangian can be written as $\mathcal{L}=\mathcal{L}_{EH}+\mathcal{L}_M$. $\mathcal{L}_{EH}$ is the Lagrangian of empty spacetime, which was indicated already in (20). $\mathcal{L}_M$ is the Lagrangian of the matter (including the electromagnetic field), which is contained in spacetime. Furthermore he assumes, that $\mathcal{L}_M$ does depend only on $(g^{\tau\sigma})$, ϕ_r, and $\mathrm{d}_\mu\phi_r$, but not on the derivatives of $(g^{\tau\sigma})$. We adopt this ansatz, and insert into the variation (21) of the integral of action the following fourth term:

$$\delta S_4=\int\limits_\omega\frac{\mathrm{d}^4x}{c}\,\delta\mathcal{L}_M=\int\limits_\omega\frac{\mathrm{d}^4x}{c}\,\frac{\sqrt{|g|}}{2}\underbrace{\Big(\frac{2}{\sqrt{|g|}}\frac{\delta\mathcal{L}_M}{\delta g^{\mu\nu}}\Big)}_{T_{\mu\nu}}\delta g^{\mu\nu}\ . \tag{68}$$

The tensor $T_{\mu\nu}$ defined this way, is called the "dynamic" energydensity-stress-tensor. Thus one eventually gets Einstein's field equation:

$$\delta S \overset{(33)}{=} \frac{c^3}{16\pi G} \int_\omega \mathrm{d}^4 x \, \sqrt{|g|} \cdot$$

$$\cdot \Big(R_{\mu\nu} - \frac{1}{2} \, g_{\mu\nu} \, (R - 2\Lambda) + \frac{8\pi G}{c^4} \, T_{\mu\nu} \Big) \, \delta g^{\nu\mu} = 0 \tag{69}$$

$$\Longrightarrow \quad R_{\mu\nu} - \frac{R}{2} \, g_{\mu\nu} + \Lambda \, g_{\mu\nu} = - \frac{8\pi \, G}{c^4} \, T_{\mu\nu} \tag{70}$$

When Einstein in 1916 published his ansatz (68), according to which the variation δS_4 of the action can be completely described by the variation $\delta g^{\mu\nu}$ of $\mathcal{L}_M$ with respect to the metric field, then he could not know that Dirac twelve years later would introduce spinor fields into physics. (68) is not correct in the case of spinor fields. Stretching the metric $g^{\mu\nu}$ is equivalent to shrinking the field amplitudes ψ and their derivatives $\mathrm{d}_\mu \psi$ in time-position-space. But the variation in spinor-space, which is necessary as well, can not be replaced by the variation $\delta g^{\mu\nu}$.

The left side of the field equation

$$R_{\mu\nu} - \frac{R}{2} \, g_{\mu\nu} + \Lambda \, g_{\mu\nu} \overset{(70)}{=} - \frac{8\pi \, G}{c^4} \, T_{\mu\nu} \tag{71}$$

is invariant under permutation of μ and ν. Consequently the same must hold for the right side. At first sight, the "dynamic" ES-tensor

$$T_{\mu\nu} \equiv \frac{2}{\sqrt{|g|}} \, \frac{\delta \mathcal{L}_M}{\delta g^{\mu\nu}} \; , \tag{72a}$$

which was defined in (68), seems to comply to that condition, because $g_{\mu\nu} = g_{\nu\mu}$ is symmetric, and $\mathcal{L}_M / \sqrt{|g|}$ is a scalar. The definition of the dynamic ES-tensor differs significantly from the definition of the canonical ES-tensors, which in case of a rigid metric (that is: in the case of Special Relativity Theory) is given by

$$\mathcal{T}_{\mu\nu} \equiv \sum_r \frac{\partial \mathcal{L}}{\partial(\mathrm{d}^\mu \phi_r)} \,\mathrm{d}_\nu \phi_r - g_{\mu\nu}\mathcal{L} \;\text{ if } g_{\mu\nu}(x) = \eta_{\mu\nu}\,\forall x \;, \tag{72b}$$

see e. g. [6, Chap. 4.2]. The sum is running over all components of all fields ϕ_r, which are contained in the Lagrangian $\mathcal{L}$. One clearly must stipulate, that the both definitions (72) of the ES-tensor match in the limit $g_{\mu\nu}(x) \rightarrow \eta_{\mu\nu}\,\forall x$. We now will investigate, whether the ES-tensors of the real Klein-Gordan field, of the electromagnetic field, and of the Dirac field, are symmetric.

(a) Real Klein-Gordan Field

The metrically covariant Lagrangian of the real Klein-Gordan field is

$$\mathcal{L}_M = \sqrt{|g|}\Big(\frac{c^2\hbar^2}{2}\, g^{\mu\nu}(\mathrm{D}_\mu\phi)\mathrm{D}_\nu\phi - \frac{m^2c^4}{2}\,\phi^2\Big) \;. \tag{73}$$

Variation of this field's action with respect to the metric field $(g^{\mu\nu})$ results into

$$\begin{aligned} \delta S = &\int\limits_\omega \frac{\mathrm{d}^4x}{c}\,\Big(\delta\sqrt{|g|}\Big)\Big(\frac{c^2\hbar^2}{2}\, g^{\mu\nu}(\mathrm{D}_\mu\phi)\mathrm{D}_\nu\phi - \frac{m^2c^4}{2}\,\phi^2\Big) + \\ &+ \int\limits_\omega \frac{\mathrm{d}^4x}{c}\,\sqrt{|g|}\,\frac{c^2\hbar^2}{2}\,\Big(\delta g^{\mu\nu}\Big)(\mathrm{D}_\mu\phi)\mathrm{D}_\nu\phi \\ \stackrel{(26)}{=} &\int\limits_\omega \frac{\mathrm{d}^4x}{c}\,\frac{\sqrt{|g|}}{2}\,\Big[- g_{\mu\nu}\frac{c^2\hbar^2}{2}\, g^{\rho\sigma}(\mathrm{D}_\rho\phi)\mathrm{D}_\sigma\phi + g_{\mu\nu}\frac{m^2c^4}{2}\,\phi^2 + \\ &+ c^2\hbar^2\,(\mathrm{D}_\mu\phi)\mathrm{D}_\nu\phi\Big]\,\delta g^{\mu\nu} = 0 \;. \end{aligned} \tag{74}$$

Comparing this with (69), one finds the dynamic ES-tensor

$$\begin{aligned} T_{\mu\nu} &= c^2\hbar^2\,(\mathrm{D}_\mu\phi)\mathrm{D}_\nu\phi - g_{\mu\nu}\Big(\frac{c^2\hbar^2}{2}\,g^{\rho\sigma}(\mathrm{D}_\rho\phi)\mathrm{D}_\sigma\phi - \frac{m^2c^4}{2}\,\phi^2\Big) \\ &\overset{(73)}{=} c^2\hbar^2\,(\mathrm{D}_\mu\phi)\mathrm{D}_\nu\phi - \frac{g_{\mu\nu}\mathcal{L}_M}{\sqrt{|g|}}\ . \end{aligned} \tag{75}$$

In the limit $g_{\mu\nu}(x)\to\eta_{\mu\nu}\,\forall x$, this is identical to the canonical ES-tensor, see e. g. [6, Chap. 7.3]. This ES-tensor is symmetric under permutation of μ and ν.

(b) Electromagnetic Field

The electromagnetic field's metrically covariant Lagrangian is

$$\mathcal{L}_M = \sqrt{|g|}\,\Big(-\frac{1}{4\mu_0}\,g^{\mu\rho}g^{\nu\sigma}F_{\rho\sigma}F_{\mu\nu}\Big)\ . \tag{76}$$

The variation of this field's action with respect to the inverse metric field $(g^{\mu\nu})$ results into

$$\begin{aligned} \delta S &= \int\limits_\omega \frac{\mathrm{d}^4x}{c}\,\Big(\delta\sqrt{|g|}\Big)\Big(-\frac{1}{4\mu_0}\,g^{\mu\rho}g^{\nu\sigma}F_{\rho\sigma}F_{\mu\nu}\Big) + \\ &+ \int\limits_\omega \frac{\mathrm{d}^4x}{c}\,\sqrt{|g|}\,\Big(-\frac{1}{4\mu_0}\,(\delta g^{\mu\rho})g^{\nu\sigma}F_{\rho\sigma}F_{\mu\nu} - \frac{1}{4\mu_0}\,g^{\mu\rho}(\delta g^{\nu\sigma})F_{\rho\sigma}F_{\mu\nu}\Big) \\ &\overset{(26)}{=} \int\limits_\omega \frac{\mathrm{d}^4x}{c}\,\frac{\sqrt{|g|}}{2}\,\Big(-\frac{1}{\mu_0}\,F_{\mu\sigma}F_\nu{}^\sigma - g_{\mu\nu}\underbrace{\Big[-\frac{1}{4\mu_0}\,F^{\rho\sigma}F_{\rho\sigma}\Big]}_{\mathcal{L}_M\,/\,\sqrt{|g|}}\Big)\,\delta g^{\mu\nu}\ . \end{aligned}$$

Comparing this with (69), one finds that the expression in the round brackets is the dynamic ES-tensor:

$$T_{\mu\nu} = -\frac{1}{\mu_0}\,F_{\mu\sigma}F_\nu{}^\sigma - \frac{g_{\mu\nu}\mathcal{L}_M}{\sqrt{|g|}} \tag{77}$$

The dynamic ES-tensor is symmetric under permutation of μ and ν. Thereby it differs from the electromagnetic field's canonical ES-tensor

$$\begin{aligned}\mathcal{T}^{\mu\nu} &= -\frac{1}{\mu_0}F^{\mu\tau}\mathrm{d}^\nu A_\tau - g^{\mu\nu}\mathcal{L} = \\ &= +\frac{1}{\mu_0}\Big((\mathrm{d}^\tau A^\mu)\mathrm{d}^\nu A_\tau - (\mathrm{d}^\mu A^\tau)\mathrm{d}^\nu A_\tau\Big) - g^{\mu\nu}\mathcal{L}\ ,\end{aligned} \tag{78}$$

as can be read for example in [6, appendix A.24]. But it is delineated in [6, appendix A.25], how the canonical ES-tensor can be converted — without changing the conserved quantities — such, that it becomes symmetric. The electromagnetic field's symmetrized canonical ES-tensor is

$$\begin{aligned}T^{\mu\nu} &= \frac{1}{\mu_0}F^{\tau\mu}\mathrm{d}^\nu A_\tau - g^{\mu\nu}\mathcal{L} - \frac{1}{\mu_0}F^{\tau\mu}\mathrm{d}_\tau A^\nu \\ &= -\frac{1}{\mu_0}F^{\mu\tau}F^\nu{}_\tau - g^{\mu\nu}\mathcal{L}\ .\end{aligned} \tag{79}$$

Thus it is identical to the dynamic ES-tensor (77) in the limit $g_{\mu\nu}(x) \rightarrow \eta_{\mu\nu}\,\forall x$.

(c) Dirac-Field

The Dirac field's (e. g. the electron-positron field's) metrically covariant Lagrangian is

$$\mathcal{L}_M = \sqrt{|g|}\;\overline{\psi}\Big(i\hbar c g^{\mu\nu}\gamma_\mu \mathrm{D}_\nu - mc^2\Big)\,\psi\ . \tag{80}$$

The variation of this field's action with respect to the inverse metric field $(g^{\mu\nu})$ results into

$$
\begin{aligned}
\delta S = &\int\limits_{\omega} \frac{\mathrm{d}^4 x}{c} \left(\delta\sqrt{|g|}\right) \overline{\psi}\Big(i\hbar c g^{\mu\nu}\gamma_\mu \mathrm{D}_\nu - mc^2\Big)\,\psi + \\
&+ \int\limits_{\omega} \frac{\mathrm{d}^4 x}{c} \sqrt{|g|}\; \overline{\psi} i\hbar c (\delta g^{\mu\nu})\gamma_\mu \mathrm{D}_\nu\,\psi \\
\overset{(26)}{=} &\int\limits_{\omega} \frac{\mathrm{d}^4 x}{c} \sqrt{|g|}\,\Big\{\, \overline{\psi}\, i\hbar c\gamma_\mu \mathrm{D}_\nu\,\psi - \\
&- \frac{g_{\mu\nu}}{2}\Big[\underbrace{\overline{\psi}\Big(i\hbar c\gamma^\alpha \mathrm{D}_\alpha - mc^2\Big)\,\psi}_{\mathcal{L}_M\,/\,\sqrt{|g|}}\Big]\Big\}\,\delta g^{\mu\nu} \;.
\end{aligned} \tag{81}
$$

Comparing this with (69), one finds that the expression in the curly brackets is the dynamic ES-tensor:

$$
T_{\mu\nu} = \overline{\psi}\, i\hbar c\gamma_\mu \mathrm{D}_\nu\,\psi - \frac{g_{\mu\nu}\mathcal{L}_M}{2\sqrt{|g|}} \tag{82}
$$

The ES-tensor is *not* symmetric under permutation of μ and ν:

$$
\overline{\psi}\,\gamma_\mu \mathrm{D}_\nu\,\psi \neq \overline{\psi}\,\gamma_\nu \mathrm{D}_\mu\,\psi \tag{83}
$$

It also is differing from the Dirac field's asymmetric canonical ES-tensor

$$
\begin{aligned}
T_{\mu\nu} &= \frac{\partial\mathcal{L}}{\partial(\mathrm{d}^\mu\overline{\psi})}\,\mathrm{d}_\nu\overline{\psi} + \frac{\partial\mathcal{L}}{\partial(\mathrm{d}^\mu\psi)}\,\mathrm{d}_\nu\psi - g_{\mu\nu}\mathcal{L} \\
&= \overline{\psi}\, i\hbar c\gamma_\mu \mathrm{d}_\nu\psi - g_{\mu\nu}\mathcal{L}
\end{aligned} \tag{84}
$$

by a factor $1/2$ in the last term.

One could simply replace the faulty dynamic ES-tensor by the symmetrized canonical ES-tensor

$$T^{\rho\sigma} = \frac{i\hbar c}{4}\Big(- (\mathrm{d}^\rho\overline{\psi})\gamma^\sigma\psi - (\mathrm{d}^\sigma\overline{\psi})\gamma^\rho\psi + \overline{\psi}\gamma^\rho\mathrm{d}^\sigma\psi + \overline{\psi}\gamma^\sigma\mathrm{d}^\rho\psi\Big) - \\ - g^{\rho\sigma}\ \underbrace{\overline{\psi}(i\hbar c\gamma^\nu\mathrm{d}_\nu - mc^2)\psi}_{\mathcal{L}}\ , \tag{85}$$

which is derived in [6, appendix A.25]. This ES-tensor obviously is symmetric under permutation of ρ and σ. But actually the problem is routed much deeper, and would be amended only superficially by that method. Fields, which have only space-time-components (like for example the Klein-Gordan field or the electromagnetic field) are invariant under a rotation of 2π around an arbitrary axis of three-dimensional position space. In contrast, the amplitudes of spinor fields (i. e. all fields with half-integer spin) change the sign under a rotation of 2π in position space, and are invariant only under rotations of 4π. That makes some basic modifications of GRT necessary, which are exceeding by far a simple correction of the ES-tensor. These modifications are described e. g. in [1].

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