Remarks on Special Relativity Theory

Contents

1. Overview

The formal content of Special Relativity Theory (SRT) can be compressed into the words: "To compute the values of physical quantities from one inertial system into an other inertial system, which is in motion relative to the first system, the Lorentz transformations must be used." Einstein started his evaluations from the "special relativity principle" and from the "principle of constancy of the speed of light", which therefore should be considered the physical formulation of Special Relativity Theory. We will demonstrate, however, that the special relativity principle is not needed for the derivation of the Lorentz transformations. Instead we will formulate a "geometry postulate", which Einstein applied implicitly, though he did not state it explicitly.

Besides the delineation of the physical assumptions, onto which Special Relativity Theory is based, we will discuss in the first, physical part of this treatise the range of validity of SRT, and the method proposed by Einstein for the synchronization of clocks.

In the second part of the treatise we will point out the conclusions, which can be drawn from the first part by merely consequent computation, without further physical considerations. First we will describe, how Einstein derived the Lorentz transformations and the invariant length of SRT from his basic assumptions. We will emphasize, that he applied only the principle of the constancy of the speed of light and — implicitly — the geometry postulate for that purpose, but not the relativity principle.

As consequences of the Lorentz transformations, we will then derive how velocities must be added according to SRT, we will describe time dilation and length contraction, and eventually discuss two of the many paradoxa of Special Relativity Theory: First the transit of a train through a tunnel, and then in very much detail the well-known twin paradox. Both turn out — like all other paradoxa of SRT — upon closer scrutiny as merely apparent inconsistencies.

2. Physical Part

2.1. Principles

Two basic assumptions are stated at the begin of Einstein's 1905 publication [1] of his Special Relativity Theory. Einstein emphasizes that they are the „Voraussetzungen, auf die sich die folgenden Überlegungen stützen“ (“preconditions, onto which the following considerations are based”). These are the two basic assumptions:

A The relativity principle: The laws, which rule the changes of physical states, are identical in two coordinate systems, which are relative to another in steady translational motion.

B The principle of the constancy of the speed of light: Any light-signal moves in the “resting” coordinate system with the certain velocity V, no matter whether that signal has been emitted by a resting or a moving body.

Actually Einstein used a plethora of further assumptions and notions, which seemed so obvious to him (and to his readers), that it seemed superfluous to mention them explicitly. That's of course inevitable. If a physical evaluation were to to start with a complete listing of all fundamental assumptions, then it would fill many thick books. And if furthermore the meaning and application range of each notion had to be clarified, that would result into an infinite regress. Hence the essential skill of a good author is to identify and highlight those few of all assumptions, which are not generally accepted as self-evident, and furthermore significant for the arguments of the evaluation.

If Einstein had by 1905 already acquired that knowledge, which he actually gained only in the following ten years in course of the development of General Relativity Theory, then he would certainly have mentioned as the third basic assumption of SRT that Euclidean geometry is valid. Here “valid” means “realizable”. It is an essential requirement, stipulated by Einstein, that coordinate systems must not only be somehow defined on the paper of the theorist, but must allow for a tangible technical realization. For that purpose, clocks are posted and read at different points of space, and the distances between the clocks are measured by rigid rulers. Cartesian coordinate systems are realized by inflexible rods. That's only possible, if the rigid rods (respectively their extensions to infinity) are realizations of Euclidean straight lines, if the sum of the inner angles of a triangle built from those rods indeed is π , if it is clearly defined what is meant if two rods are said to be adjusted "parallel".

By 1905 it seemed still self-evident that Euclidean geometry can (at least in principle) be realized by stiff rods. Today we know, thanks to Einstein's later evaluations, that this is possible only under certain conditions, only within limited space areas, and in any case only approximately. Furthermore we learned that approximations to Euclidean straight lines can much easier be realized by light rays than by stiff rods. Therefore we state this additional basic assumption of Special Relativity Theory:

C The geometry postulate: Light rays are realizations of the straight lines of Euclidean geometry, which has been described by Euclid [2] and defined more precisely by Hilbert [3].

The formulation of assumption **B** is redundant. If the velocity of all light-signals has the identical value [1] V , then V is of course independent of the speed of the source. Einstein doubly emphasizes that fact, because in those years there was much discussion about a hypothesis due to Walter Ritz, who had noticed that the results of the interference experiments of Michelson and Morley could alternatively be explained by the assumption, that the speed of light does depend on the speed of the light source. Ritz' hypothesis — which anyway would have produced lots of new problems — has been disproved in the following years due to astronomical observations. Therefore by today any further discussion of that hypothesis seems superfluous.

[1] In this article, Einstein's letter V for the speed of light in vacuum will be replaced by the today commonly used letter c.

It is a further weakness of Einstein's formulation of assumption **B**, that he speaks of a "resting" coordinate system. That's prone to misunderstandings. It is obvious from the context of his article that he is convinced, that the speed of light in vacuum has the identical value c in *any* inertial system. Therefore we chose this simpler and clearer formulation:

B' The principle of the constancy of the speed of light: An electromagnetic signal in vacuum has the identical velocity c in any arbitrary inertial system.

Assumptions of the type **A**,**B'**,**C** can be neither derived nor proved. Einstein found them by guessing, of course being guided into the right direction by the experimental facts. This is obvious in case of assumption **B'** , but less clear with regard to assumption **A**. The only method to check such assumptions is to derive conclusions from them, which could be experimentally disproved in case that they should be wrong. If an assumption and the conclusions derived from it have stood many such checks without getting falsified, then our confidence into it's validity is strengthened, and we call that assumption a law of nature. A positive "proof" (in the strict meaning of that word) of the correctness of a law of nature is not possible, however, given that we can only perform a finite number of attempts of falsification.

We will demonstrate immediately that from the assumptions **B'** and **C** the Lorentz transformations can be derived. Therefore these transformations — but not the Galilei transformations — must be applied in all parts of physics, in which those fundamental assumptions are valid, i. e. not only in electrodynamics, but also in mechanics.

If the Lorentz transformations are applied in computations for the conversion of physical quantities between inertial systems which are in relative motion, then regularly consistent results of theory and experiments are found.

Einstein made no use at all of the relativity principle (assumption **A**) in his derivation of the Lorentz transformations. Assumptions **B'** and **C** alone are sufficient for that purpose. When Einstein placed a relativity principle at the begin of the publication of Special Relativity Theory, this illustrates his motivation and considerations in those days. But nowhere in his treatise any formal conclusion is drawn from that principle. His derivation of the Lorentz transformation is exclusively based on assumption **B'** and (only implicitly) on assumption **C**, as will be demonstrated in detail in section 3.1 . The relativity principle **A** is *no* presupposition of Special Relativity Theory.[2]

[2] Only many months after I had completed this little article, I became aware of a remark, which Einstein made eleven years after he published SRT . In the first paragraphs of an article on General Relativity Theory [4] he writes:

"The special theory of relativity is based on the following postulate, which is also satisfied by the mechanics of Galileo and Newton. If a system of coordinates K is chosen so that, in relation to it, the physical laws hold good in their simplest form, *the same* laws also hold good in relation to any other system of coordinates K' moving in uniform translation relatively to K . This postulate we call the 'special principle of relativity'. The word 'special' is meant to intimate that the principle is restricted to the case when K' has a motion of *uniform translation* relatively to K , but that the equivalence of K' and K does not extend to the case of *non-uniform* motion of K' relatively to K . Thus the special theory of relativity does not depart from classical mechanics through the postulate of relativity, but exclusively through the postulate of the constancy of the velocity of light in vacuo, from which, in combination with the special principle of relativity, there follow, in the well-known way, the relativity of simultaneity, the Lorentz transformation, and the related laws for the behavior of moving rigid bodies and clocks."

So this is what Einstein wanted to say: Special Relativity Theory is based on the one hand onto the principle **B'** of the constancy of the speed of light, and on the other hand onto Newton's classical mechanics and Maxwell's classical electrodynamics. Integral parts of these two classical theories are the principle **A** of special relativity, and the assumption **C** of the validity of Euclidean geometry, which thus are parts of the "self-evident" basic

2.2. Inertial systems

"Every body perseveres in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed thereon", writes Newton in the Principia [5, Axioms, or laws of motion; Law 1]. As external "forces impressed thereon" he considers gravity, pressure, push, magnetic and electric forces, ... By today we would say: Gravitative, electro-weak, or strong interactions. Forces, which are no external forces, are inertial forces. In his General Relativity Theory, Einstein will consider inertial and gravitative forces as identical ("equivalence principle"). But in SRT he still sticks to Newton's notion of force, according to which gravity is a force impressed externally onto a body, and must be distinguished from inertial forces.

If in a coordinate system all bodies, which are not subject to external forces, are in a "state of rest, or of uniform motion in a right line," then that coordinate system is by definition an *inertial system*. In contrast, if in a coordinate system a body moves accelerated, even though no external force is acting on the body, then that coordinate system is by definition an *accelerated system*, and the force, which causes the acceleration of the body, is called *inertial force*.

Einstein does nowhere in [1] use the notion "inertial system". In his formulation of the relativity principle, he merely is speaking of "two coordinate systems, which are relative to another in a state of uniform translational motion". From the context it is obvious, however, that the SRT is about relative to another moving inertial systems, not about accelerated systems. At the begin of part I

assumptions, which need not necessarily be mentioned explicitly. It would be really interesting to know, whether Einstein considered this explanation a correction of his 1905 publication, or merely a clarification of a quite unclear formulation.

§ 1 of his article, Einstein specifies “a coordinate system, in which Newton’s mechanical equations hold.” By this restriction Einstein certainly wants to specify that there are no inertial forces acting in this coordinate system. If inertial forces are included, then Newton’s mechanics are valid in arbitrary coordinate systems, and Einstein’s formulation wouldn’t have any sensible meaning. Furthermore it becomes implicitly visible in the sequel of his article (see e. g. footnote 6 on page 14), that Einstein is considering inertial systems.

2.3. The application range of SRT

In the principle **B’** of the constancy of the speed of light there is no restriction with regard to the point of time, or the place, or the direction of the light signal. At any time, at any place, and in any direction, an electromagnetic signal moves in vacuum with the always identical velocity c. That does not imply, however, that time and space as well are homogeneous. Remember that c is the quotient of the distance which the light travels, and the time interval which passes by during that travel. Therefore, if the space should be shrunk somewhere, then the light velocity could still be unchanged, if only the time is shrunk accordingly in that range of space, i. e. if clocks are slower in that range of space than in other places.

If light rays would enter from outside such a shrunk space range, they would be refracted according to Fermat’s principle. Hence these light rays wouldn’t be realizations of Euclidean straight lines, in contradiction to the geometry postulate, assumption **C**. By today, we know gravitational lenses, and we know that light rays are refracted when they pass by near the sun’s surface. Thus space is indeed not homogeneous, but more or less shrunk at various places. Consequently SRT is not valid everywhere, but only at those places, where — at least approximately — it's preconditions **B'** *and* **C** both are valid. **B'** is everywhere and at any time precisely correct, according to today knowledge. In contrast, **C** is merely an approximation, which holds nowhere strictly with perfect precision. Consequently the application range of SRT is defined by **C**: Special Relativity Theory holds with that precision, with which it's precondition **C** is correct.

In a range of space, in which SRT holds, there are no gravitational lenses. Gravitational lenses exist everywhere, where the space is inhomogeneous and/or anisotropic. Hence space is homogeneous and isotropic everywhere, where the SRT is valid. In that case time, being the quotient of the homogeneous and isotropic space and of the always and everywhere identical speed of light, must as well be homogeneous.

2.4. How to synchronize clocks

At first sight, the principle **B'** of the constancy of the speed of light seems incompatible with countless observations within classical mechanics. On the other hand, the experiments of Michelson and Morley seem to imply exactly this assumption. Einstein understood that the discrepancies between **B'** and the well-known facts of mechanics vanish, if moving rulers are shorter than rulers at rest, and if moving clocks are slower than clocks at rest. Following Einstein's reasoning, we therefore first of all must clarify how space, time, and velocity have to be measured correctly. Thereby we must in particular pay attention to the state of motion or rest of rulers and clocks.

Distances between points of space *at rest* must according to Einstein as always be measured such, that rigid rulers (e. g. copies of the prototype meter bar) are concatenated in a continuous row, and the minimum number of rulers is counted, which are needed to bridge the distance between two points r_a and r_b. Equivalently we could in principle replace Einstein's rigid rulers by this method:[3] We emit at point r_a an electromagnetic signal in direction of point r_b. At r_b the signal is reflected by a mirror back towards r_a. The time t_{aba}, which the signal needs to travel from r_a to r_b and back, is measured by a clock which is at rest at point r_a. As the speed of the signal according to assumption **B'** is c, the time of travel is

$$t_{aba} = \frac{2 \cdot D}{c} \quad \text{with } D \equiv \text{distance between } r_a \text{ and } r_b\ . \tag{1}$$

Thus the distance between r_a and r_b is[3]

$$D = \frac{1}{2} \cdot \frac{t_{aba}}{\text{second}} \cdot 299\,792\,458\,\text{meter}\ . \tag{2}$$

Likewise simple is the measurement of time. Time is measured by counting periodic events, e. g. the oscillations of the balance spring in a wrist-watch, or[4] the oscillations of a micro-wave resonator which is adjusted to the 9.192 631 770 GHz transition in the ground state of ^{133}Cs. Again we must not forget this important restriction: The clock must be at rest in the used coordinate system.

Problems turn up if we want to measure velocities. The mean velocity v of an object, which starts at time t_0 to move from space-point r_a in direction of r_b, and arrives there at time t_1, is

$$v = \frac{D}{t_1 - t_0} \quad \text{with } D \equiv \text{distance between } r_a \text{ and } r_b\ . \tag{3}$$

[3] According to the actually (2010) valid definition [6], the meter is the length of the distance, which light in vacuum travels in (1/299 792 458) seconds.

[4] According to the actually (2010) valid definition [6], 1 second is 9 192 631 770 times the period-duration of the micro-wave radiation, which corresponds to the transition between the hyperfine-structure levels in the ground state of ^{133}Cs.

This is a problem for the following reason: We want to measure the time, at which an event happens, by a clock which is at rest at the place of the event. The start of the object at place r_a is an event, and the object's arrival at r_b is a second event. And we have the suspicion — which will be confirmed in the sequel — that moving clocks are slower than clocks at rest. Hence the time t_0 must be measured with a clock which is at rest at place r_a, and the time t_1 must be measured with a clock which is at rest at r_b. The problem is not to place one clock at r_a and another clock at r_b. The problem is that those two clocks must be synchronized, if we want to get a reasonable value of the velocity. We can not achieve the synchronization, if we assemble the clocks at one place, synchronize them, and then transport them to r_a and r_b respectively, because any transport of the clocks will disturb the synchronization. By 1905, Einstein could conclude this fact only indirectly. Since the seventies of the 20th century the direct experimental proof is possible, and has been reproduced repeatedly.

If we alternatively try to measure both events, the start at r_a and the arrival at r_b, with the same clock — for example the clock which is resting at r_a —, then we need an additional signal, which reports the event from point r_b to r_a, and a theory of the travel time which that signal needs. But don't we have such a signal with known transmission speed c readily available in form of light signals? No, we merely know (or guess) that c is a well-defined constant of nature, but we do *not* know [5] it's value. The number 299 792 458 showing up in footnote 3 is part of the meter's *definition*, but not a measured value. Still this consideration guides us in the right direction. The method, which Einstein [1] suggested for the synchronization of two clocks, which are at rest at the places

[5] Of course we know the value of the speed of light approximately, but our evaluation here is about the principle!

r_a and r_b (i. e. which are neither moving relative to one another, nor relative to the inertial system), is quite similar to the method for the measurement of distances:

When the clock at r_a displays the time t_0, a light signal is emitted from r_a towards r_b. Due to a mirror at r_b the signal is reflected towards r_a. At the moment of reflection, the clock resting at r_b displays the time t_1. The reflected signal is detected at r_a. In the moment of detection, the clock resting at r_a displays the time t_2.

As the signal transverses the distance D between r_a and r_b with equal velocity on it's forward and return run according to the principle of constancy of the speed of light (assumption **B'**) — i. e. with the identical velocity c in any reference system — , the signal's

$$\text{travel time} = \frac{\text{distance}}{\text{velocity}}$$

must as well be identical in both directions, i. e.

$$t_1 - t_0 = \frac{D}{c} = t_2 - t_1 \tag{4a}$$

must hold. To synchronize the clocks, they therefore must be adjusted to

$$t_1 = \frac{1}{2}(t_2 + t_0) \ . \tag{4b}$$

Note that furthermore a definition of the notion "same time" results from this method: An event with space-time coordinates (t_a, x_a, y_a, z_a) happens *by definition* at the same time as an event with space-time coordinates (t_b, x_b, y_b, z_b), if clocks, which are synchronized according to (4), display the same time at both space-time points.

Note that the synchronization is defined only within one inertial system, but not between two inertial systems which are in relative motion. Consequently the notion "same time" is as well defined only within one inertial system, but not between two inertial systems which are in relative motion. We will see that two events, which happen at same time in one inertial system, do *not* happen at the same time in an other inertial system, which is in motion relative to the first one, see page 30.

3. Conclusions

3.1. Derivation of the Lorentz transformations from assumptions B' and C

We want to demonstrate that the Lorentz transformations can be derived from the two postulates

B' The principle of the constancy of the speed of light: An electromagnetic signal in vacuum has the identical velocity c in any arbitrary inertial system.

C The geometry postulate: Light rays are realizations of the straight lines of Euclidean geometry, which has been described by Euclid [2] and defined more precisely by Hilbert [3].

without utilization of an additional "relativity principle". Thereby we follow closely Einstein's arguments [1]. Thus Einstein did not apply the "relativity principle" in the derivation of the Lorentz transformations.

Let an inertial system, marked by a dash$'$, move into an arbitrary direction of three-dimensional space relatively to another inertial system, which has no dash-mark in our notation.

In the un-dashed system we define a rectangular cartesian coordinate system such, that it's x-axis is parallel to (and pointing into the same direction as) the relative velocity v of the dashed coordinate system. Thus the components of the velocity v of the dashed system, as measured in the un-dashed system, are

$$(v_x, v_y, v_z) = (v, 0, 0)\ . \tag{5}$$

The y- and z-axis of the un-dashed coordinate system, and it's origin, are chosen arbitrarily.

We define a further rectangular cartesian coordinate system, which is at rest in the dashed system. It's origin is chosen arbitrarily. It's x'-axis is directed parallel to (and pointing into the same directions as) the un-dashed x-axis, and the y'- and z'-axis are chosen parallel to (and pointing into the same direction as) the y'- and z'-axis, respectively.[6]

We now want to find out, how the coordinate-quadruple (t, x, y, z) can be computed from (t', x', y', z'), and vice versa. For that purpose, Einstein first evaluated the dashed time $t'(t, x, y, z)$ as a function of the un-dashed coordinates.

Let a light-signal be emitted at space-time point

$$(t'_0, x'_0, y'_0, z'_0) = (t_0, x_0, y_0, z_0) \tag{6}$$

parallel to (and in the same direction as) the positive x'- resp. x-axis. At (t'_M, x'_M, y'_0, z'_0) the signal arrives at a mirror, which reflects it back into direction of the source. The mirror is fixed in the dashed system, i. e. it is moving in the un-dashed system with velocity v in x-direction.

In the un-dashed system, the mirror has at time t_0 the coordinates (t_0, x_M, y_0, z_0). It's coordinates at an arbitrary time t are

[6] At this point Einstein implicitly made use of the fact that both coordinate systems are inertial systems. Otherwise the space would be not euclidean, and the coordinate axes could not be aligned "parallel", because the notion "parallel" would not even be defined. These implication were of course not yet known to Einstein by 1905 .

$(t, x_M + v(t - t_0), y_0, z_0)$. The light signal arrives at the mirror at time t_M of the un-dashed system. At that moment, the mirror's x-coordinate is $x_M + v(t_M - t_0)$. The velocity c of the signal is identical in both systems according to assumption **B'**. The distance, which the light must travel in the un-dashed system from the source to the mirror, is larger than $x_M - x_0$, because the mirror is moving continuously away from the source. Larger distance and identical velocity results into longer travel time. The time, which the signal needs in the un-dashed system for the distance from the source to the mirror, is

$$\begin{aligned} t_M - t_0 &= \frac{x_M + v(t_M - t_0) - x_0}{c} \\ c(t_M - t_0) - v(t_M - t_0) &= x_M - x_0 \\ t_M - t_0 &= \frac{x_M - x_0}{c - v} \ . \end{aligned} \tag{7}$$

Consequently

$$t_M \overset{(7)}{=} t_0 + \frac{x_M - x_0}{c - v} \tag{8a}$$

$$v(t_M - t_0) \overset{(7)}{=} v\,\frac{x_M - x_0}{c - v} \ , \tag{8b}$$

and the signal arrives at the mirror at this space-time point:

$$\begin{aligned} (t'_M, x'_M, y'_0, z'_0) &= (t_M, x_M + v(t_M - t_0), y_0, z_0) = \\ &\overset{(8)}{=} \left(t_0 + \frac{x_M - x_0}{c - v}, x_M + v\frac{x_M - x_0}{c - v}, y_0, z_0 \right) \end{aligned} \tag{9}$$

At (t'_A, x'_0, y'_0, z'_0) the reflected signal arrives at a detector with fixed position in the dashed system, and is registered. In the un-dashed system, the distance between mirror and detector is shorter than $x_M - x_0$, because the detector moves with velocity v

in opposing direction to the reflected signal. Therefore we get for the return path of the signal in the denominator $c+v$, where we had $c-v$ for the forward path in (9). Thus the signal arrives at the detector at space-time point

$$\begin{aligned}(t'_A, x'_0, y'_0, z'_0) &= (t_A, x_0 + v(t_A - t_0), y_0, z_0) = \\ &= \Bigg(t_0 + \frac{x_M - x_0}{c-v} + \frac{x_M - x_0}{c+v}, \\ &\qquad x_0 + v\left(\frac{x_M - x_0}{c-v} + \frac{x_M - x_0}{c+v}\right), y_0, z_0\Bigg) \ . \end{aligned} \tag{10}$$

In the sequel we want to find out the function $t'(t,x,y,z)$. We know that function already for three space-time points:

$$t'_0 \overset{(6)}{=} t'(t_0, x_0, y_0, z_0) \tag{11a}$$

$$t'_M \overset{(9)}{=} t'\left(t_0 + \frac{x_M - x_0}{c-v}, x_M + v\frac{x_M - x_0}{c-v}, y_0, z_0\right) \tag{11b}$$

$$\begin{aligned} t'_A \overset{(10)}{=} t'\Bigg(&t_0 + \frac{x_M - x_0}{c-v} + \frac{x_M - x_0}{c+v}, \\ &x_0 + v\left(\frac{x_M - x_0}{c-v} + \frac{x_M - x_0}{c+v}\right), y_0, z_0\Bigg) \end{aligned} \tag{11c}$$

It is typical for Einstein, that he did not try to indicate the general function $t'(t,x,y,z)$ immediately, but first derived the differential quotients $\frac{\partial t'}{\partial t}, \frac{\partial t'}{\partial x}, \frac{\partial t'}{\partial y}, \frac{\partial t'}{\partial z}$. He applied for that purpose a quite tricky method. He “tottered” infinitesimally at the mirror, i. e. he evaluated the derivative $\frac{\partial t'}{\partial x_M}$:

$$\frac{\partial t'}{\partial x_M} = \frac{\partial t'}{\partial t}\frac{\partial t}{\partial x_M} + \frac{\partial t'}{\partial x}\frac{\partial x}{\partial x_M} + \frac{\partial t'}{\partial y}\frac{\partial y}{\partial x_M} + \frac{\partial t'}{\partial z}\frac{\partial z}{\partial x_M} \tag{12a}$$

$$\frac{\partial t'_0}{\partial x_M} \overset{(12a),(11a)}{=} 0 \tag{12b}$$

$$\frac{\partial t'_M}{\partial x_M} \overset{(12a),(11b)}{=} \frac{\partial t'}{\partial t}\frac{1}{c-v} + \frac{\partial t'}{\partial x}\left(1+\frac{v}{c-v}\right) \tag{12c}$$

$$\frac{\partial t'_A}{\partial x_M} \overset{(12a),(11c)}{=} \frac{\partial t'}{\partial t}\left(\frac{1}{c-v}+\frac{1}{c+v}\right) + \frac{\partial t'}{\partial x}\left(\frac{v}{c-v}+\frac{v}{c+v}\right) \tag{12d}$$

Now Einstein used (12), to compute the derivative of the synchronization condition of the clocks in the dashed system with respect to x_M :

$$\frac{\partial t'_M}{\partial x_M} \overset{(4b)}{=} \frac{\partial}{\partial x_M}\left(\frac{1}{2}(t'_A+t'_0)\right)$$

$$\frac{\partial t'}{\partial t}\frac{1}{c-v} + \frac{\partial t'}{\partial x}\left(1+\frac{v}{c-v}\right) \overset{(12c),(12d),(12b)}{=}$$

$$= \frac{1}{2}\frac{\partial t'}{\partial t}\left(\frac{1}{c-v}+\frac{1}{c+v}\right) + \frac{1}{2}\frac{\partial t'}{\partial x}\left(\frac{v}{c-v}+\frac{v}{c+v}\right)$$

$$\frac{1}{2}\frac{\partial t'}{\partial t}\left(\frac{1}{c-v}-\frac{1}{c+v}\right) = \frac{1}{2}\frac{\partial t'}{\partial x}\left(\frac{v}{c+v}-2-\frac{v}{c-v}\right)$$

$$\frac{\partial t'}{\partial t}\left(\frac{c+v-c+v}{c^2-v^2}\right) = \frac{\partial t'}{\partial x}\left(\frac{v(c-v)-2(c^2-v^2)-v(c+v)}{c^2-v^2}\right)$$

$$-\frac{v}{c^2}\frac{\partial t'}{\partial t} = \frac{\partial t'}{\partial x} \tag{13}$$

Thereby we have found the differential quotients $\frac{\partial t'}{\partial t}$ and $\frac{\partial t'}{\partial x}$. Remarkably they are mutually dependent ! To find as well $\frac{\partial t'}{\partial y}$ and $\frac{\partial t'}{\partial z}$, we send a light signal from space-time point

$$(t'_0, x'_0, y'_0, z'_0) = (t_0, x_0, y_0, z_0) \tag{14}$$

in direction of the y'-axis towards a mirror, which is fixed in the dashed system at point (t', x'_0, y'_M, z'_0). The time-dependent coordinates of this point in the un-dashed system are $(t, x_0 + v(t - t_0), y_M, z_0)$. Making use of assumption **B'**, we insert c for the signal's velocity in both coordinate systems. Note that again the runtime $t_M - t_0$ of the signal from the source to the mirror is larger in the un-dashed system than $(y_M - y_0)/c$, because the mirror is moving in x-direction while the signal is underway. Hence the signal must run a longer "inclined" path, and it's runtime $t_M - t_0$ from the source to the mirror can be computed by Pythagoras' theorem:

$$\begin{aligned} c^2(t_M - t_0)^2 &= (y_M - y_0)^2 + v^2(t_M - t_0)^2 \\ t_M - t_0 &= \frac{y_M - y_0}{\sqrt{c^2 - v^2}} \end{aligned} \tag{15}$$

Thus the signal's space-time coordinates are at the moment of arrival at the mirror

$$\begin{aligned} &(t'_M, x'_0, y'_M, z'_0) = \\ &\quad = (t_0 + \frac{y_M - y_0}{\sqrt{c^2 - v^2}}, x_0 + v(t_0 + \frac{y_M - y_0}{\sqrt{c^2 - v^2}}), y_M, z_0) \ . \end{aligned} \tag{16}$$

The mirror reflects the signal, which then is registered by a detector which is fixed at (t', x'_0, y'_0, z'_0) in the dashed system. The signal's space-time coordinates in the moment of detection are

$$(t'_A, x'_0, y'_0, z'_0) = (t_0 + 2\frac{y_M - y_0}{\sqrt{c^2 - v^2}}, x_0 + v(t_0 + 2\frac{y_M - y_0}{\sqrt{c^2 - v^2}}), y_0, z_0). \tag{17}$$

Again Einstein "tottered" infinitesimally at the mirror to evaluate the derivative of t' with respect to y_M:

$$\frac{\partial t'_0}{\partial y_M} = 0 \tag{18a}$$

$$\frac{\partial t'_M}{\partial y_M} = \frac{\partial t'}{\partial t}\frac{1}{\sqrt{c^2 - v^2}} + \frac{\partial t'}{\partial x}\frac{v}{\sqrt{c^2 - v^2}} + \frac{\partial t'}{\partial y} \tag{18b}$$

$$\frac{\partial t'_A}{\partial y_M} = \frac{\partial t'}{\partial t}\frac{2}{\sqrt{c^2 - v^2}} + \frac{\partial t'}{\partial x}\frac{2v}{\sqrt{c^2 - v^2}} \tag{18c}$$

Using (18), the derivative of the synchronization condition (4b) with respect to y_M can be computed:

$$\frac{\partial}{\partial y_M}t'_M \overset{(4b)}{=} \frac{\partial}{\partial y_M}\left(\frac{1}{2}(t'_A+t'_0)\right)$$
$$\frac{\partial t'}{\partial t}\frac{1}{\sqrt{c^2-v^2}}+\frac{\partial t'}{\partial x}\frac{v}{\sqrt{c^2-v^2}}+\frac{\partial t'}{\partial y} \overset{(18b),(18c),(18a)}{=}$$
$$=\frac{1}{2}\left(\frac{\partial t'}{\partial t}\frac{2}{\sqrt{c^2-v^2}}+\frac{\partial t'}{\partial x}\frac{2v}{\sqrt{c^2-v^2}}\right)$$
$$\frac{\partial t'}{\partial y}=0 \tag{19}$$

By the same method we find

$$\frac{\partial t'}{\partial z}=0\ . \tag{20}$$

Now we know the derivatives of t' with respect to the un-dashed coordinates t,x,y,z . It is easy to check (by insertion) that

$$t'-t'_0=\gamma\left(t-t_0-\frac{v(x-x_0)}{c^2}\right) \tag{21}$$

fulfills the equations (13), (19), and (20). Here γ is a still unknown function, which does neither depend on t,x,y,z nor on x_M nor on y_M , because otherwise (21) wouldn't be a solution of (13), (19), and (20). But γ may depend on v , and we will find that this is indeed the case. The constants t'_0 , t_0 , and x_0 , are fixed by the boundary condition $(t'_0,x'_0,y'_0,z'_0)=(t_0,x_0,y_0,z_0)$.

With (21) we have found the transformation of the time coordinates (with γ still to be determined). To construct the transformation of the space coordinates, Einstein again made use of assumption **B'**, according to which the velocity c of a light signal in vacuum is identical in both systems. From space-time point

$(t'_0, x'_0, y'_0, z'_0) = (t_0, x_0, y_0, z_0)$ we send a light signal in x-direction ($= x'$-direction). The signal's x'-coordinate at time t' is

$$x' = c(t' - t'_0) + x'_0 \ , \tag{22a}$$

and it's x-coordinate at time t is

$$x = c(t - t_0) + x_0 \ . \tag{22b}$$

Inserting (22) into (21) gives

$$\begin{aligned} \frac{x' - x'_0}{c} &= \gamma\Big(\frac{x - x_0}{c} - \frac{vc(t - t_0)}{c^2}\Big) \\ x' - x'_0 &= \gamma\Big(x - x_0 - v(t - t_0)\Big) \ . \end{aligned} \tag{23}$$

Now we send from space-time point $(t'_0, x'_0, y'_0, z'_0) = (t_0,x_0,y_0,z_0)$ a light signal in y'-direction. The signal's y'-coordinate at time t' is

$$y' = c(t' - t'_0) + y'_0 \ , \tag{24}$$

and it's x-coordinate at time t is

$$x = v(t - t_0) + x_0 \ . \tag{25a}$$

As the signal runs an "inclined" path in the un-dashed system with velocity c, we get by means of Pythagoras' theorem for it's y-coordinate

$$y - y_0 = \sqrt{c^2(t - t_0)^2 - (x - x_0)^2} \overset{(25a)}{=} (\sqrt{c^2 - v^2})(t - t_0) \ . \tag{25b}$$

Inserting (24) and (25a) into (21) gives

$$\frac{y'-y'_0}{c}=\gamma\Big(t-t_0-\frac{v^2(t-t_0)}{c^2}\Big)\ . \tag{26}$$

From this result we get due to insertion of (25b)

$$\frac{y'-y'_0}{c}=\gamma\Big(\frac{y-y_0}{\sqrt{c^2-v^2}}-\frac{v^2(\frac{y-y_0}{\sqrt{c^2-v^2}})}{c^2}\Big)$$

$$y'-y'_0=\gamma\Big(\sqrt{1-\frac{v^2}{c^2}}\Big)(y-y_0)\ . \tag{27}$$

By the same method we get

$$z'-z'_0=\gamma\Big(\sqrt{1-\frac{v^2}{c^2}}\Big)(z-z_0)\ . \tag{28}$$

This is a comprehensive overview of the four transformations, which we have found:

$$t'-t'_0\overset{(21)}{=}\gamma\left(t-t_0-\frac{v(x-x_0)}{c^2}\right) \tag{29a}$$

$$x'-x'_0\overset{(23)}{=}\gamma(x-x_0-v(t-t_0)) \tag{29b}$$

$$y'-y'_0\overset{(27)}{=}\gamma\left(\sqrt{1-\frac{v^2}{c^2}}\right)(y-y_0) \tag{29c}$$

$$z'-z'_0\overset{(28)}{=}\gamma\left(\sqrt{1-\frac{v^2}{c^2}}\right)(z-z_0) \tag{29d}$$

with $(t'_0,x'_0,y'_0,z'_0)=(t_0,x_0,y_0,z_0)$

with γ: still unknown, see (31)!

Our final task is to determine γ. This is, how Einstein did it: Using (29), he transformed the dashed coordinates into a double-dashed coordinate system, which is moving with velocity v' into the x'-direction of the dashed system. It's x''-axis is parallel to (and pointing into the same direction as) the x'-axis, it's y''-axis is parallel to (and pointing into the same direction as) the y'-axis, and it's z''-axis is parallel to (and pointing into the same direction as) the z'-axis:

$$t'' - t''_0 \overset{(29a)}{=} \gamma \left(t' - t'_0 - \frac{v'(x' - x'_0)}{c^2} \right)$$

$$\overset{(29b)}{=} \gamma^2 \left(t - t_0 - \frac{v(x - x_0)}{c^2} - \frac{v'(x - x_0 - v(t - t_0))}{c^2} \right)$$

$$= \gamma^2 \left((t - t_0) \left(1 - \frac{-v'v}{c^2} \right) - \frac{(v + v')(x - x_0)}{c^2} \right) \quad (30a)$$

$$x'' - x''_0 \overset{(29b)}{=} \gamma (x' - x'_0 - v'(t' - t'_0))$$

$$\overset{(29a)}{=} \gamma^2 \left(x - x_0 - v(t - t_0) - v' \left(t - t_0 - \frac{v(x - x_0)}{c^2} \right) \right)$$

$$= \gamma^2 \left((x - x_0) \left(1 - \frac{-v'v}{c^2} \right) - (v + v')(t - t_0) \right) \quad (30b)$$

$$y'' - y''_0 \overset{(29c)}{=} \gamma \left(\sqrt{1 - \frac{v'^2}{c^2}} \right) (y' - y'_0)$$

$$\overset{(29c)}{=} \gamma^2 \left(\sqrt{1 - \frac{v'^2}{c^2}} \right) \left(\sqrt{1 - \frac{v^2}{c^2}} \right) (y - y_0) \quad (30c)$$

$$z'' - z''_0 \overset{(29d)}{=} \gamma \left(\sqrt{1 - \frac{v'^2}{c^2}} \right) (z' - z'_0)$$

$$\overset{(29d)}{=} \gamma^2 \left(\sqrt{1 - \frac{v'^2}{c^2}} \right) \left(\sqrt{1 - \frac{v^2}{c^2}} \right) (z - z_0) \quad (30d)$$

with $(t''_0, x''_0, y''_0, z''_0) = (t'_0, x'_0, y'_0, z'_0) = (t_0, x_0, y_0, z_0)$

Now Einstein chose $v' = -v$. With this choice, the double-dashed coordinates must be identical to the un-dashed coordinates, because this is just the back-transformation from the dashed into the un-dashed system. Thus we get

$$\gamma^{-1} = \sqrt{1 - \frac{v^2}{c^2}} \;, \tag{31}$$

and (29) become the well-known Lorentz transformations:

$$t' - t'_0 = \frac{t - t_0 - v(x - x_0)/c^2}{\sqrt{1 - v^2/c^2}} \tag{32a}$$

$$x' - x'_0 = \frac{x - x_0 - v(t - t_0)}{\sqrt{1 - v^2/c^2}} \tag{32b}$$

$$y' - y'_0 = y - y_0 \tag{32c}$$

$$z' - z'_0 = z - z_0 \tag{32d}$$

$$\text{with } (t'_0, x'_0, y'_0, z'_0) = (t_0, x_0, y_0, z_0)$$

Next we want to derive the invariant of SRT. For that purpose, we multiply (32a) by the constant factor c, and square it. We square (32b) as well. Then we subtract the second equation from the first:

$$\begin{aligned} c^2(t' - t'_0)^2 - (x' - x'_0)^2 \;&= \\ = \gamma^2\Big(t - t_0 - \frac{v}{c^2}(x - x_0)\Big)^2 &- \gamma^2\Big(x - x_0 - v(t - t_0)\Big)^2 \\ = c^2(t - t_0)^2 \underbrace{\gamma^2\Big(1 - \frac{v^2}{c^2}\Big)}_{1} &- (x - x_0)^2 \underbrace{\gamma^2\Big(1 - \frac{v^2}{c^2}\Big)}_{1} \end{aligned} \tag{33a}$$

Furthermore

$$(y' - y_0')^2 \overset{(32c)}{=} (y - y_0)^2 \tag{33b}$$

$$(z' - z_0')^2 \overset{(32d)}{=} (z - z_0)^2 \ . \tag{33c}$$

The equations (33) hold, if the dashed coordinate system is moving parallel to the x-axis of the un-dashed system. It is plausible that the general rule

$$\begin{aligned} c^2(t' - t_0')^2 - (x' - x_0')^2 - (y' - y_0')^2 - (z' - z_0')^2 &= \\ = c^2(t - t_0)^2 - (x - x_0)^2 - (y - y_0)^2 - (z - z_0)^2 \end{aligned} \tag{34}$$

holds for relative movements of the two coordinate systems in any arbitrary direction. We skip the formal proof. (34) is the fundamental invariant of Special Relativity Theory. Minkowski[7] suggested to consider (34) as the invariant distance square in a four-dimensional space, called "space-time". While in three-dimensional euclidean space the distance square $(x-x_0)^2+(y-y_0)^2+(z-z_0)^2$ of the points (x, y, z) and (x_0, y_0, z_0) is invariant and identical in any reference system, in the four-dimensional space of SRT the square distance $c^2(t-t_0)^2-(x-x_0)^2-(y-y_0)^2-(z-z_0)^2$ of the points (ct, x, y, z) and (ct_0, x_0, y_0, z_0) is invariant and identical in any reference system. The Minkowski space of SRT is no euclidean space, because the squares in (34) do not all have the same signs. Instead by (34) a hyperbolic metric is defined.

By multiplying (32a) with the constant factor c, and combining the components of (32) to four-vectors, (32) can be written in matrix form:

$$\begin{pmatrix} ct' - ct_0' \\ x' - x_0' \\ y' - y_0' \\ z' - z_0' \end{pmatrix} = \Lambda_{x\text{-boost}} \begin{pmatrix} ct - ct_0 \\ x - x_0 \\ y - y_0 \\ z - z_0 \end{pmatrix} \tag{35}$$

[7] Hermann Minkowski (1864 - 1909)

The transformation

$$\Lambda_{x\text{-boost}} \equiv \begin{pmatrix} \frac{1}{\sqrt{1-(v/c)^2}} & \frac{-v/c}{\sqrt{1-(v/c)^2}} & 0 & 0 \\ \frac{-v/c}{\sqrt{1-(v/c)^2}} & \frac{1}{\sqrt{1-(v/c)^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{36a}$$

is called "Lorentz-boost" in x-direction. General Lorentz transformations may be boosts in arbitrary directions and/or pure rotations in three-dimensional space. Without proof we indicate the Lorentz transformations for rotations of the coordinate system around the z-axis and around the x-axis:

$$\Lambda_{z\text{-rot}} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & \sin\phi & 0 \\ 0 & -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{36b}$$

$$\Lambda_{x\text{-rot}} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\psi & \sin\psi \\ 0 & 0 & -\sin\psi & \cos\psi \end{pmatrix} \tag{36c}$$

Any arbitrary rotation in three-dimensional space can be synthesized by combination of a rotation around the z-axis, then a rotation around the new x-axis, and eventually a rotation around the new z-axis. A Lorentz-boost in arbitrary direction can be synthesized by a rotation of the x-axis into the direction of the planned boost, then the boost by means of (36a), and eventually the back-rotation of the axes. Thus any arbitrary Lorentz transformation can be synthesized by (repeated) combinations to the three transformations (36).

The Lorentz transformations are consequences of the two basic assumptions **B'** and **C** . We may state: **B'** and **C** are the physical formulation of Special Relativity Theory, while the transformations (36) in combination with the invariant (34) — which is a consequence of those transformations — are the mathematical formulation of SRT.

The relativity principle **A** is no precondition of SRT (Einstein made no use of it at any point of his derivation of the Lorentz transformations), but a correct conclusion from it.

3.2. How to add velocities

Let a dashed coordinate system move relatively to an un-dashed system with velocity v in x-direction. Let a double-dashed coordinate system move relatively to the dashed system with velocity v' in x'-direction. Let x- and x'-direction be identical. Then the double-dashed coordinate system is moving relatively to the un-dashed system with velocity v'' in x-direction. According to Newton's mechanics, $v'' = v' + v$. According to Special Relativity Theory, however, velocities have to be added differently, i. e. $v'' \neq v' + v$. We are now going to evaluate, how velocities are to be added according to SRT.

The Lorentz transformations (32) become for the double-dashed, the dashed, and the un-dashed system:

$$t'' - t''_0 = \frac{t' - t'_0 - v'(x' - x'_0)/c^2}{\sqrt{1 - v'^2/c^2}} =$$

$$= \frac{t - t_0 - v(x - x_0)/c^2 - v'(x - x_0 - v(t - t_0))/c^2}{\sqrt{1 - v^2/c^2} \cdot \sqrt{1 - v'^2/c^2}} \tag{37a}$$

$$x'' - x''_0 = \frac{x' - x'_0 - v'(t' - t'_0)}{\sqrt{1 - v'^2/c^2}} =$$

$$= \frac{x - x_0 - v(t - t_0) - v'(t - t_0 - v(x - x_0)/c^2)}{\sqrt{1 - v^2/c^2} \cdot \sqrt{1 - v'^2/c^2}} \tag{37b}$$

$$y'' - y''_0 = y' - y'_0 = y - y_0 \tag{37c}$$

$$z'' - z''_0 = z' - z'_0 = z - z_0 \tag{37d}$$

with $(t''_0, x''_0, y''_0, z''_0) = (t'_0, x'_0, y'_0, z'_0) = (t_0, x_0, y_0, z_0)$

Alternatively, we can with (32) indicate directly the Lorentz transformations of the space-time coordinates from the un-dashed to the double-dashed system:

$$t'' - t''_0 = \frac{t - t_0 - v''(x - x_0)/c^2}{\sqrt{1 - v''^2/c^2}} \tag{38a}$$

$$x'' - x''_0 = \frac{x - x_0 - v''(t - t_0)}{\sqrt{1 - v''^2/c^2}} \tag{38b}$$

$$y'' - y''_0 = y - y_0 \tag{38c}$$

$$z'' - z''_0 = z - z_0 \tag{38d}$$

with $(t''_0, x''_0, y''_0, z''_0) = (t_0, x_0, y_0, z_0)$

Now we divide (37a) by (37b):

$$\begin{aligned}\frac{t'' - t''_0}{x'' - x''_0} &= \frac{t - t_0 - v(x - x_0)/c^2 - v'(x - x_0 - v(t - t_0))/c^2}{x - x_0 - v(t - t_0) - v'(t - t_0 - v(x - x_0)/c^2)} = \\ &= \frac{(t - t_0)(c^2 + v'v)/c^2 - (v + v')(x - x_0)/c^2}{(x - x_0)(c^2 + v'v)/c^2 - (v + v')(t - t_0)} = \\ &= \frac{t - t_0 - (x - x_0)(v + v')/(c^2 + v'v)}{x - x_0 - (t - t_0)c^2(v + v')/(c^2 + v'v)}\end{aligned} \tag{39}$$

And we divide (38a) by (38b):

$$\frac{t'' - t''_0}{x'' - x''_0} = \frac{t - t_0 - (x - x_0)v''/c^2}{x - x_0 - (t - t_0)v''} \tag{40}$$

Comparing (40) and (39) we get immediately

$$v'' = \frac{v + v'}{1 + v'v/c^2} \; . \tag{41}$$

Newton's result $v'' = v' + v$ is a good approximation to (41) for small velocities $(vv'/c)^2 \ll 1$. More interesting is the addition of large velocities. For $v = v' = 0.9c$ we get

$$v'' = \frac{0.9c + 0.9c}{1 + 0.81} \approx 0.9945c \neq 1.8c\ . \tag{42}$$

No matter how close the velocities v and v' approach the velocity c of light in vacuum, their relativistic sum is always smaller than c.

3.3. Time dilation, length contraction

Let the dashed coordinate system move with velocity v in x-direction of the un-dashed system. A clock at rest at point x'_A, y'_A, z'_A in the dashed system has at time t'_A the space-time coordinates (t'_A, x'_A, y'_A, z'_A), and at time t'_B the space-time coordinates (t'_B, x'_A, y'_A, z'_A). At both space-time points there are in addition clocks, which are at rest in the un-dashed system. Thus in total we are working with three clocks. The two un-dashed clocks are synchronized as described in section 2.4, and their coordinates are(t_A, x_A, y_A, z_A) respectively (t_B, x_B, y_B, z_B) at the moments when they meet the dashed clock. Consequently the time interval between the two points of time is according to (32a)

$$\begin{aligned} t'_B - t'_A &= \frac{t_B - t_A - v(x_B - x_A)/c^2}{\sqrt{1 - v^2/c^2}} \\ &= \frac{t_B - t_A - v(x_A + v(t_B - t_A) - x_A)/c^2}{\sqrt{1 - v^2/c^2}} \\ &= (t_B - t_A)\sqrt{1 - v^2/c^2} \quad \leq (t_B - t_A)\ , \end{aligned} \tag{43}$$

„woraus folgt, dass die Angabe der [bewegten] Uhr (im ruhenden System betrachtet) pro Sekunde um $(1 - \sqrt{1 - (v/V)^2})$ Sek. [...]

zurückbleibt.“ [8] writes Einstein [1].

The slowing-down of moving clocks is called time dilation. As the velocity of light c, being the quotient of a distance and a time interval, is identical in any reference system, there must exist some phenomenon of space shrinking, which correlates with the slowing-down of clocks. That phenomenon is called length contraction:

Consider a stiff rod of length L' in the dashed coordinate system, which is at rest in that system. One end of the rod has at any time t' the coordinates (t', x'_A, y'_A, z'_A), the other end the coordinates (t', x'_B, y'_B, z'_B). To measure the rod's length L in the un-dashed system (in which the rod is moving with velocity v in x-direction), we must measure the positions of it's both ends at the same time, say at time t_m. At that point of time, the coordinates of the rod's ends are (t_m, x_A, y_A, z_A) and (t_m, x_B, y_B, z_B). Using (32), we can immediately indicate the components of L' resp. L:

$$L'_x = x'_B - x'_A = \frac{x_B - x_A - v(t_m - t_m)}{\sqrt{1 - v^2/c^2}} = \frac{L_x}{\sqrt{1 - v^2/c^2}}$$

$$L_x = L'_x\sqrt{1 - v^2/c^2} \quad \leq L'_x \tag{44a}$$

$$L_y = y_B - y_A = y'_B - y'_A = L'_y \tag{44b}$$

$$L_z = z_B - z_A = z'_B - z'_A = L'_z \tag{44c}$$

Thus $L' > L$ if $v \neq 0$ and $L_x > 0$. L' is the rod's length as measured in the coordinate system in which the rod is at rest. L is the rod's length as measured in the coordinate system in which the rod is moving. The length of the moving rod is smaller than the length of the rod at rest, even though we are of course considering at any time the identical rod! How is that possible? We get the essential hint from a colleague who is at rest relatively to the rod,

[8] “which implies that the display of the [moving] clock (as seen in the resting system) is lagging by $(1 - \sqrt{1 - (v/V)^2})$ sec. [...] per second.”

i. e. at rest in the dashed system: “You have measured the positions of the two ends of the rod at different points of time.” Therefore lets consider again the point of time t_m of the measurement by means of equation (32a):

$$t'_{mA} - t'_0 = \frac{t_m - t_0 - v(x_A - x_0)/c^2}{\sqrt{1 - v^2/c^2}}$$

$$t'_{mB} - t'_0 = \frac{t_m - t_0 - v(x_B - x_0)/c^2}{\sqrt{1 - v^2/c^2}}$$

$$t'_{mA} - t'_{mB} = \frac{-v(x_A - x_B)/c^2}{\sqrt{1 - v^2/c^2}} \quad \begin{cases} < 0 \text{ if } x_A > x_B \\ > 0 \text{ if } x_A < x_B \end{cases} \tag{45}$$

Indeed, according to a clock resting in the dashed system, we measured the position of the rod’s leading end too early and/or the position of the rod’s rear end too late. Thus the difference of the measurement results is caused by the definition of “same time” in section 2.4 . That definition was a consequence of the synchronization method described there, and that was again a consequence of assumption **B’**, the principle of the constancy of the speed of light. Hence in the end that principle is the cause, why events, which happen at same time in one reference system, are happening at different times in a reference system which is moving relatively to the other system.

Einstein writes [1]: „Ein starrer Körper, welcher in ruhendem Zustande ausgemessen die Gestalt einer Kugel [mit dem Radius R] hat, hat also in bewegtem Zustande – vom ruhenden System aus betrachtet – die Gestalt eines Rotationsellipsoides mit den Achsen

$$R\sqrt{1 - \frac{v^2}{V^2}}, R, R \ .$$

Während also die Y- und Z-Dimension der Kugel (also auch jedes starren Körpers von beliebiger Gestalt) durch die Bewegung nicht modifiziert erscheinen, erscheint die X-Dimension im Verhältnis $1 : \sqrt{1 - v^2/V^2}$ verkürzt, also um so stärker, je grösser v ist. Für $v = V$ schrumpfen alle bewegten Objekte – vom ‚ruhenden' System aus betrachtet – in flächenhafte Gebilde zusammen.“ [9] Here Einstein uses V for the speed of light in vacuum, for which we have used the notation c.

Einstein's wording “seems to be shortened” is prone to the misunderstanding, that the length contraction might be something like a parallax error or an optical illusion. But moving objects are really — in any sensible meaning of that word — contracted versus the same objects at rest. That's similar to the question whether the people in New Zealand are standing heads-up, while the Europeans are standing heads-down, or vice versa. Notions like “up” and “down” do not have a universally valid meaning, and can be sensibly used only in relation to a locally defined reference system. The Relativity Theory got it's name, because this theory clarified that many notions like e. g. “length” of a rod, which in the past seemed to have a universally valid meaning, can actually be uniquely defined only relatively to a certain reference system. The measured length L is relatively to the un-dashed coordinate system as real and correct as the length L' relatively to the dashed system.

[9] “A stiff body, which has the form of a sphere [with radius R] if measured in the state of rest, consequently has in the state of movement — as seen from the resting system — the form of a rotation-ellipsoid with axes

$$R\sqrt{1 - \frac{v^2}{V^2}}, R, R \ .$$

While consequently the sphere's (and consequently also any arbitrary other stiff body's) Y- and Z-dimension seem not to be modified by the movement, the X-dimension seems to be shortened by the ratio $1 : \sqrt{1 - v^2/V^2}$, i. e. the stronger, the larger v is. For $v = V$ all moving objects shrink — as seen from the ‘resting’ system — to area-like formations.”

3.4. Space-time diagrams

The relations between relatively moving coordinate systems can be displayed most clearly by means of space-time diagrams. As an example, we consider a dashed and an un-dashed coordinate system. The axes of the two systems are oriented into the same directions, and the origin of the dashed system is moving relatively to the un-dashed system with velocity v in direction of the un-dashed x-axis, see figure 1 . (This diagram has been constructed with $v = c/2$.)

The un-dashed system is displayed with rectangular axes. The units of the x-axis and of the ct-axis are chosen such, that the green sketched worldline of a light signal, which passed at time $t = 0$ the point $x = 0$ in x-direction, is inclined by 45° versus the both blue coordinate axes. The horizontal blue lines are lines of same time in the un-dashed system, and the vertical blue lines are

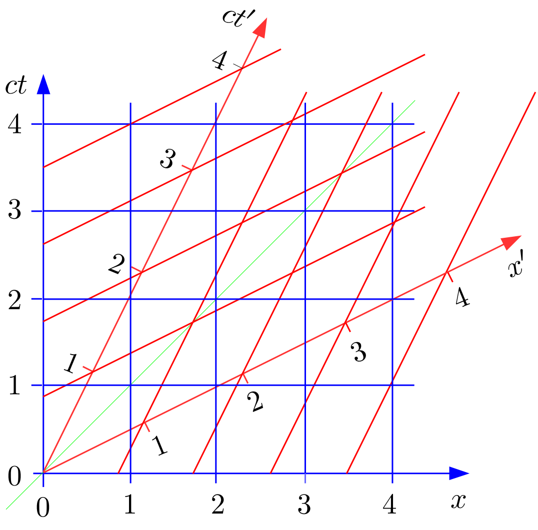

Fig. 1: A space-time diagram

lines of same position in this system.

This relation holds for the angle between the ct-axis and the ct'-axis:

$$\tan(ct, ct') = \frac{x}{ct} = \frac{v}{c} \tag{46a}$$

It fixes the direction (but not the scale) of the ct'-axis.

The steep red lines, which are parallel to the ct'-axis, are lines of same position in the dashed coordinate system. According to Newton's mechanics, the horizontal blue lines would be lines of same time as well in the dashed system, i. e. the x'-axis would be parallel to the x-axis. According to SRT, however, the direction of the x'-axis must be determined by means of the Lorentz transformations. The x'-axis are the points $(t' = 0, x')$. Setting furthermore $t_0 = 0$ and $x_0 = 0$, on this axis holds

$$\begin{aligned} t' = 0 &\overset{(32a)}{=} \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}} \\ \Longrightarrow \quad \frac{ct}{x} &= \frac{v}{c} = \tan(x, x') \overset{(46a)}{=} \tan(ct, ct') \ . \end{aligned} \tag{46b}$$

Thus the x'-axis is rotated versus the x-axis by the same angle towards the light signal's worldline, as the ct'-axis versus the ct-axis. This relation always holds: If the scales of the axes of a space-time diagram are chosen such, that the worldline of a light signal in vacuum has the same inclination versus the space axis and the time axis, then the same holds for any arbitrary system, whose origin is moving with constant (positive or negative) velocity relatively to the first system.

The flat red lines, which are parallel to the x'-axis, are lines of same time in the dashed coordinate system. They are crossing the horizontal blue lines of same time in the un-dashed system. Events, which happen at same time in one system, therefore do not happen at same time in the other system.

By (46) we have fixed the directions of the ct'-axis and the x'-axis, but not yet their scales. First we consider the x'-axis. According to (44a), the x-component L'_x of the length of an object at rest in the dashed system is contracted to L_x in the un-dashed system:

$$L_x \overset{(44a)}{=} L'_x\sqrt{1-v^2/c^2} \quad \leq L'_x \tag{47}$$

Lets use the same units — say, meter — on the x-axes of both systems. An object with length $L'_x = 1\,\mathrm{m}$ at rest in the dashed system is in the un-dashed system only $\sqrt{1-v^2/c^2}\,\mathrm{m}$ long. Consequently a straight line, which is parallel to the ct'-axis and intersects the x-axis at that point which is representing $\sqrt{1-v^2/c^2}\,\mathrm{m}$, intersects the x'-axis at that point which is representing $1\,\mathrm{m}$.

The graphic fig. 1 on page 32 has been constructed with $v = c/2$. Therefore the steep red line, which intersects the x-axis at $\sqrt{1-c^2/(4c^2)} = 0.866$ is intersecting the x'-axis at 1.

As the ct'-axis and the x'-axis enclose the same angle with the green world-line of the light signal (see fig. 1), the lengths of the scale-parts must be identical on both axes. Thus we have found the length of the scale-parts of both dashed axes.

Note the perfect symmetry: The steep red line through scale-part n on the x'-axis in fig. 1 intersects the x-axis left of scale-part n on that axis. And the vertical blue line through scale-part n on the x-axis intersects the x'-axis left of scale-part n on that axis: Objects at rest in the dashed system are contracted by the factor $\sqrt{1-v^2/c^2}$ in the un-dashed system, and objects at rest in the un-dashed system are contracted by the factor $\sqrt{1-v^2/c^2}$ in the dashed system.

The flat red line through scale-part n on the ct'-axis intersects the ct-axis below scale-part n on that axis. And the horizontal blue line through scale-part n on the ct-axis intersects the ct'-axis below scale-part n on that axis: Clocks at rest in the dashed system are slower by the factor $\sqrt{1-v^2/c^2}$ in the un-dashed system, and clocks at rest in the un-dashed system are slower by the factor $\sqrt{1-v^2/c^2}$ in the dashed system.

3.5. A train racing through a tunnel

The strange phenomena of length contraction and time dilation have prompted the invention of many paradoxa, in which Special Relativity Theory seems to be self-contradicting. Until today, however, no paradox has been found, which did not turn out upon accurate analysis as a mere pseudo problem. Concluding this article, we now are going to discuss two of the best known paradoxa.

The first paradox is about a train, which is racing with half the speed of light in vacuum through a tunnel. The tunnel has gates at it's both ends, and a mechanism secures that only one of the two gates can be open at any time. Let the length of the tunnel be exactly 100 m in the system in which it is at rest. The length of the train, in the system in which it is at rest, is as well exactly 100 m. Viewed in the rest system of the tunnel, which will be called the un-dashed system in the sequel, there seems to be no problem: Due to it's huge velocity of $v=c/2$, the train's length in that system is only

$$\text{length}_{\text{train}} \overset{(44a)}{=} 100\,\text{m}\sqrt{1-\frac{v^2}{c^2}} = 86.6\,\text{m} . \tag{48}$$

Thus the train can completely enter the tunnel, while the entrance gate is opened and the exit gate is closed. Then the entrance gate can be closed and the exit gate can be opened, before the head of the train arrives at the exit gate. Seen from the dashed rest-system of the train, however, there seems to be no chance for a passage of the train without crash with at least one of the doors: In the dashed system, the tunnel is racing with half the velocity of light towards the train, which is at rest. Hence the tunnel's length is only 86.6 m , while the train's length is 100 m . Consequently head and rear of the train will both protrude from the tunnel for some time during the train's passage. Thus it seems impossible that at any time minimum one of the gates can be closed.

The train's collision with one of the doors is an objective fact. Whether that fact does happen or not happen can not depend on the coordinate system which is applied for the description of the process. By means of the space-time diagram displayed below, its easy to see that actually the train can pass the tunnel without collision with a gate.

The tunnel is at rest in the un-dashed blue system. The train is at rest in the dashed red system. In both systems one x-unit resp. one x'-unit is 100 m. The pale yellow marked tunnel extends from

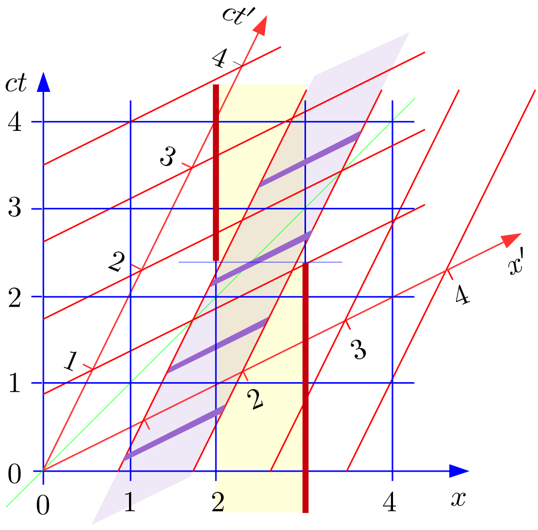

Fig. 2: Train and tunnel

$x = 2$ to $x = 3$. The closed exit gate is indicated by a fat brown line at $x = 3$. At time $ct \approx 2.4$, marked by a horizontal blue line, the entrance gate (fat brown line at $x = 2$) gets closed, and the exit gate gets opened.

The head of the train is at $x' = 2$ in it's red rest system, it's rear end at $x' = 1$. The range occupied by the train is marked by pale blue coloration. For four certain points of time the train's position is indicated by violet bars. At $ct' \approx 1.3$ the train indeed is protruding at the same time from both ends of the tunnel. But "at same time" only in the dashed coordinate system! Measured by clocks, which are at rest in the undashed system, the train is protruding from the entrance gate only well before $ct \approx 2.4$, and only well after $ct \approx 2.4$ from the tunnel's exit gate. The catastrophe can be evaded, because the notion "same time" is meaning something different in the two coordinate systems.

3.6. The twin paradox

The twins Mike and Charley want to know it precisely. They decide for an experimental test of time dilation. On their 20$^{\text{th}}$ birthday, each of them enters a spaceship, and accelerates within 3 months (as measured with their board clocks) to $0.98c$ (measured in a coordinate system fixed to earth) in direction of the galactic center. Then Charley shuts down his engine and continues (with no acceleration) to glide with $0.98c$, while Mike accelerates within 6 months (as measured with his board clock) to $0.98c$ (measured in a coordinate system fixed to earth) in direction back to earth. Then Mike accelerates within 3 months (as measured with his board clock) to approximately zero (measured in a coordinate system fixed to earth), and lands after one year (as measured with his board clock) of total flight duration, on his 21$^{\text{st}}$ birthday, gently on earth.

Charley continues for 5 years (as measured with his board clock) to glide forward with $0.98c$ (measured in a coordinate system fixed to earth), before he accelerates within 6 months (as measured with his board clock) to $0.98c$ (measured in a coordinate system fixed to earth) back in earth direction. He continues to glide without acceleration for 5 years (as measured with his board clock) with $0.98c$ (measured in a coordinate system fixed to earth) in earth direction. Then he accelerates within 3 months (as measured with his board clock) to about zero (measured in a coordinate system fixed to earth), and lands after 11 years (as measured with his board clock) of total flight duration gently on earth. Its his 31^{st} birthday! Since many months he has looked forward to the reunion with Mike.

As Charley moved with his spaceship for many years very fast — with $0.98c$, to be precise — relatively to earth and to Mike, Mike must have become quite old by now, because according to the time-dilation formula (43) Charley's fast moving board-clock was slowed down significantly in comparison to clocks resting on earth. Correspondingly, Charley's life processes and aging, which as well are some type of clock, should have proceeded significantly less than those of Mike, who lived at rest on earth during those years.

We could, however, take the opposite point of view: In a reference system in which Charley's spaceship is at rest, the earth — including Mike — moved for 5 years with $0.98c$ away from Charley's spaceship, and later for 5 years with $0.98c$ in direction towards Charley's spaceship. As Mike was for 2×5 years in very fast motion relative to Charley, he should at Charley's return be significantly younger than Charley.

The earth may be considered in good approximation an inertial system. Charley's spaceship is as well an inertial system as long as it is not accelerated by it's engine. There is no preferred inertial system in SRT, all inertial systems are equally valid. Hence both points of view, the one according to which Mike should be younger at Charley's return, and the one according to which Charley at his return should be younger than Mike, seem to be equally valid. A mind-boggling result. For that reason, the experiment performed by Mike and Charley is well-known as the "twin paradox".

We get no information from SRT about the impact of the acceleration phases, which lasted in total for one year (according to the board-clocks), because during these phases the spaceships were accelerated systems, not inertial systems. SRT is exclusively about inertial systems.[10] But exactly that is the reason, why Mike made his short space trip. Each of the twins experienced exactly the same acceleration phases, and spent one year of his respective life under identical accelerations. Hence that time can have no impact on the net difference of age of the twins. If there is a difference of age between the brothers at Charley's return, then that difference can only be caused by the acceleration-free phases. And the effect of those phases can be computed by means of SRT.

Using (43), lets compute what result is to be expected, if the earth is considered a resting system, and Charley's spaceship a moving system:

$$\begin{aligned} \text{time}_{\text{spaceship}} &= \text{time}_{\text{earth}} \cdot \sqrt{1-(0.98c)^2/c^2} \\ 10\,\text{years}_{\text{spaceship}} &\approx \text{time}_{\text{earth}} \cdot 0.2 \\ &\approx 50\,\text{years}_{\text{earth}} \cdot 0.2 \end{aligned} \tag{49}$$

Mike should have grown 50 years older while Charley grew 10 years older, according to this computation. At the day of Charley's return, his 31^{st} birthday, Mike should be about 71 years old.

[10] Remark added in 2019: This is not exactly correct. The acceleration phases of the twin paradox *can* be treated by SRT, see the "addendum in July 2019" at the end of this section.

What result do we get, if we consider Charley's spaceship a coordinate system at rest?

$$\begin{aligned} \text{time}_{\text{earth}} &= \text{time}_{\text{spaceship}} \cdot \sqrt{1 - (0.98c)^2/c^2} \\ &\approx (2 \cdot 5\,\text{years}) \cdot 0.2 \\ &\approx 2\,\text{years} \end{aligned} \tag{50}$$

According to this computation, Mike should have grown only 2 years older, while Charley grew 10 years older. Hence at the day of Charley's return, his 31$^{\text{st}}$ birthday, Mike should be about 23 years old.

71 or 23 years? Charley is extremely curious, when he crawls out of the airlock of his spaceship. A hale pensioner steps to him and embraces him heartily. Its the 71 year old Mike.

Mike is glad that he can explain the result of the experiment to his young twin brother. He has found in the meantime in a second-hand bookstore an age-old yellowed issue [7] of the "Circulars" of the Astrophysical Institute Neunhof, dated May 2010, in which the facts are explained:

The situation seems paradoxical, because there seems to be perfect symmetry: First the spaceship moves with $0.98c$ away from earth. Or the earth moves with $0.98c$ away from the spaceship. Then the earth moves with $0.98c$ towards the spaceship. Or the spaceship moves with $0.98c$ towards the earth. Only on second sight the asymmetry becomes visible: Actually we are using in total 3 inertial systems, not only 2 .

We define the name S_{R1} for that inertial system, in which Charley's spaceship is at rest during the first part of his journey, and the name S_{R2} for that system, in which the spaceship is at rest during the second part of Charley's trip. For the inertial system, in which the earth — including Mike — is at rest, we define the name S_E . When we considered the earth a resting system, we computed the complete experiment in system S_E and got the correct result. When we considered Charley's spaceship a resting system, then we first described the experiment in system S_{R1}, and then made a flying change to S_{R2}. When changing the reference system, we must take care to avoid some possible pitfalls.

To understand what went wrong when we changed the reference system, we now will evaluate the complete experiment, from begin to end, consistently in the inertial system S_{R1}. During the first part of his trip, Charley is at rest in this system for 5 years, and thereby grows 5 years older. In the same time interval, Mike grows

$$\begin{aligned} \text{time}_{\text{earth}} &= 5\,\text{years}\,\sqrt{1-(0.98c)^2/c^2} \\ &\approx 5\,\text{years}\cdot 0.2 \\ &\approx 1\,\text{year} \end{aligned} \tag{51}$$

older on earth. Then Charley changes the direction of his journey, and travels with $0.98c$ (as measured in the system S_E) towards the earth. Measured in S_{R1}, the earth is moving with $0.98c$ in the same direction as the spaceship. The spaceship's velocity (measured in S_{R1}) is of course *not* $1.96c$. We must add the velocities relativistically correctly. According to (41), the velocity of Charley's spaceship (measured in S_{R1}) is

$$v = \frac{0.98c + 0.98c}{1 + 0.98^2} \approx 0.99980c\,. \tag{52}$$

This part of the flight lasts 5 years (as measured by Charley's board clock), in which Charley grows 5 years older. Let's compute the length of that time interval in the system S_{R1}:

$$\begin{aligned} 5\,\text{years} &= \text{time}_{S_{R1}}\sqrt{1-(0.99980c)^2/c^2} \\ &\approx \text{time}_{S_{R1}}\cdot 0.0202 \\ &\approx 247\,\text{years}\cdot 0.0202 \end{aligned} \tag{53}$$

Meanwhile Mike grows older on earth by

$$\begin{aligned} \text{time}_{\text{earth}} &= 247\,\text{years}\,\sqrt{1-(0.98c)^2/c^2} \\ &\approx 247\,\text{years}\cdot 0.2 \\ &\approx 49\,\text{years}\ . \end{aligned} \tag{54}$$

Thus Mike in total grows about 50 years older, while Charley grows 10 years older. At Charley's 31st birthday, Mike is about 71 years old.

Again we arrived at the correct result. And now it is clearly visible, what went wrong in the incorrect computation (50). According to his board clock, Charley spent 5 years traveling without acceleration in one direction, and later 5 years traveling without acceleration in the return direction. In the coordinate system S_{R1} the first part of the journey took 5 years, and the return travel took 247 years. Hence Charley spent (according to clocks at rest in S_{R1}) about 2% of the total travel time with the first part of the flight, and about 98% with the return flight. Due to the symmetry of the setup we know — even if we don't exercise that computation explicitly — that he spent (according to clocks which are at rest in S_{R2}) about 98% of the total travel time with the first part of his flight, and about 2% with the return flight. Thus we have considered in our wrong computation the first 2% of the total journey in S_{R1}, and computed how old Mike grew during that time. Then we changed to system S_{R2} and considered the return travel, i. e. in that system the last 2% of the travel, and computed how old Mike grew during that time. 96% of the travel time, and Mike's aging during that time, have slipped our attention due to the incautious change of reference systems. If we multiply the wrong result (50) = 2 years with the correction factor $(1-96\%)^{-1}$, then we get again the correct value of 50 years.

Addendum in July 2019

On page 39 I wrote: "We get no information from SRT about the impact of the acceleration phases, [. . .] because [. . .] SRT is exclusively about inertial systems." This is not exactly correct.

In an article [8], published in 2019, Pepino and Mabile reminded their readers (including me) that the acceleration phases can be treated like this, using exclusively the formalism of SRT:

The time interval $t_{\text{B,Charley}} - t_{\text{A,Charley}}$ between events A and B happening on Charley's space ship as measured by his board clock is related to the time interval $t_{\text{B,Earth}} - t_{\text{A,Earth}}$ as measured by clocks at rest on Earth due to

$$t_{\text{B,Charley}} - t_{\text{A,Charley}} \overset{(43)}{=} (t_{\text{B,Earth}} - t_{\text{A,Earth}})\sqrt{1 - v^2/c^2} \ , \qquad (55)$$

if Charley is moving with the constant velocity v relative to Earth. During the acceleration phases, v isn't constant. But we can extend (55) to the acceleration phases due to

$$t_{\text{B,Charley}} - t_{\text{A,Charley}} \overset{(55)}{=} \int_{t_{\text{A,Earth}}}^{t_{\text{B,Earth}}} \mathrm{d}t_{\text{Earth}} \sqrt{1 - v^2(t_{\text{Earth}})/c^2} \qquad (56)$$

with a time-dependent velocity $v(t_{\text{Earth}})$. At start of Charley's space ship, his board clock is synchronized to the clock on earth, and we set both clocks to zero:

$$t_{\text{start,Charley}} = t_{\text{start,Earth}} = 0 \qquad (57)$$

Defining B as the event, when Charley first time switches off his engines after 3 months = 0.25 years on his board clock, we get

$$0.25\,\text{years}_{\text{Charley}} \overset{(56)}{=} \int_{0}^{t_{\text{B,Earth}}} \mathrm{d}t_{\text{Earth}} \sqrt{1 - v^2(t_{\text{Earth}})/c^2} \ . \qquad (58)$$

We did not yet fix whether

$$\frac{\mathrm{d}v}{\mathrm{d}t_{\mathrm{Charley}}} = \frac{0.98c}{0.25\,\mathrm{years}_{\mathrm{Charley}}} = \mathrm{constant}_{\mathrm{C}} \tag{59a}$$

$$\text{or} \quad \frac{\mathrm{d}v}{\mathrm{d}t_{\mathrm{Earth}}} = \frac{0.98c}{t_{\mathrm{B,Earth}}} = \mathrm{constant}_{\mathrm{E}} \tag{59b}$$

or whatever. As the ratio $\mathrm{d}t_{\mathrm{Earth}}/\,\mathrm{d}t_{\mathrm{Charley}}$ is not constant in course of the acceleration, (59a) and (59b) can not both be realized. To make our computation as simple as possible, we now decide arbitrarily for (59b). With this specification, during the first acceleration phase

$$v(t_{\mathrm{Earth}}) = \frac{0.98c}{t_{\mathrm{B,Earth}}} \cdot t_{\mathrm{Earth}} \ . \tag{60}$$

Consequently

$$0.25\,\mathrm{years}_{\mathrm{Charley}} \overset{(58),(60)}{=} \int\limits_0^{t_{\mathrm{B,Earth}}} \mathrm{d}t_{\mathrm{Earth}} \sqrt{1 - 0.98^2\,\frac{t_{\mathrm{Earth}}^2}{t_{\mathrm{B,Earth}}^2}} \ .$$

With the substitution

$$x = \frac{t_{\mathrm{Earth}}}{t_{\mathrm{B,Earth}}} \quad , \quad \mathrm{d}x = \frac{\mathrm{d}t_{\mathrm{Earth}}}{t_{\mathrm{B,Earth}}} \tag{61}$$

this results into

$$\begin{aligned} 0.25\,\mathrm{years}_{\mathrm{Charley}} &= t_{\mathrm{B,Earth}} \int\limits_0^1 \mathrm{d}x\,\sqrt{1-0.98^2x^2} = \\ &= t_{\mathrm{B,Earth}} \cdot \left[\frac{x}{2}\sqrt{1-0.98^2x^2} - \frac{1}{2\cdot 0.98}\cdot \arcsin(-0.98x)\right]_0^1 = \\ &= t_{\mathrm{B,Earth}} \cdot 0.7987 \ . \end{aligned} \tag{62}$$

Here the integral number 245 from [9] has been used. Thus the first acceleration phase, which lasted 0.25 years according to Charley's board clock, lasted $(0.25/0.7987)$ years $= 0.313$ years on a clock at rest on Earth. And the total of all acceleration phases, which took 1 year on Charley's board clock, took $4 \cdot 0.313$ years $= 1.252$ years on a clock at rest on Earth.

For the time of flight with no acceleration, we have solved the integral (56) already in (49). Thus, if Mike had not made his short space trip, but had stayed at rest on earth during all Charley's trip, his age at Charley's return (Charley's 31st birthday) would be $(20 + 1.252 + 50)$ years $= 71.252$ years.

References

[1] A. Einstein: *Zur Elektrodynamik bewegter Körper*, Ann. Phys. (Leipzig) **17**, 891 – 920 (1905), http://dx.doi.org/10.1002/andp.19053221004
english translation: http://einsteinpapers.press.princeton.edu/vol2-trans/154

[2] Euclid: *Elements*, Ed.: J. L Heiberg (greek text) (B. G. Teubner, Leipzig, 1883 – 1916)
english translation: R. Fitzpatrick, Univ. Texas, 2007
http://farside.ph.utexas.edu/Books/Euclid/Elements.pdf

[3] David Hilbert: *Grundlagen der Geometrie* (B. G. Teubner, Stuttgart, 1899) http://www.archive.org/details/grunddergeovon00hilbrich

[4] A. Einstein : *Die Grundlage der allgemeinen Relativitätstheorie* , Ann. Phys. (Leipzig) **49**, 769 – 822 (1916)
http://dx.doi.org/10.1002/andp.19163540702
english translation: http://einsteinpapers.press.princeton.edu/vol6-trans/158

[5] Isaac Newton : *Philosophiae Naturalis Principia Mathematica* (Guil. & Joh. Innys, London, 3rd ed. 1726)
http://dx.doi.org/10.3931/e-rara-1235 , English translation: https://archive.org/details/newtonspmathema00newtrich or https://en.wikisource.org/wiki/The_Mathematical_Principles_of_Natural_Philosophy_(1846)

[6] Bureau International de Poids es Mesure:
SI Brochure: The International System of Units (SI)
publication available at https://www.bipm.org

[7] *Remarks on Special Relativity Theory*
APIN Circular se05211 (2010)
http://www.astrophys-neunhof.de/mtlg/se05211.pdf

[8] R. A. Pepino, R. W. Mabile : *Pedagogical Materials and Suggestions to Cure Misconceptions Connecting Special and General Relativity* , arXiv 1907.02497 (2019)
https://arxiv.org/abs/1907.02497

[9] I. N. Bronstein, K. A. Semendjajew :
Taschenbuch der Mathematik
(Harry Deutsch, Zürich und Frankfurt/Main, 1973)