Electromagnetic Units

In electrodynamics, various systems of units are commonly used, in which not only different numerical values, but different physical dimensions are assigned to quantities like charges or fields. We describe in this article, how the formulas of electrodynamics can be translated simply and efficiently in-between the various systems of units.

According to the definition of the meter, the speed of light in vacuum is

$$c \equiv 2.997\,924\,58 \cdot 10^8\,\frac{\text{m}}{\text{s}}\ . \tag{1}$$

In addition we will define immediately five further constants $\mu_0, \epsilon_0, e, b, r$. μ_0 is the magnetic field-constant of the vacuum, and ϵ_0 is the electric field-constant of the vacuum. The constants e, b, r are related to the electric field $\boldsymbol{E}$, the magnetic field $\boldsymbol{B}$ (sometimes called induction), and the density of electric charge ρ, respectively. These five constants are defined differently in the various systems of units. The definitions differ not only in their numerical values, but also in their physical dimensions. Consequently, transformations inbetween the different systems of units are much more intricate than for example the transformation from "pounds per square inch" into "Newton per square-meter". Here the dimensions of both specifications are "pressure"; only their numerical values differ. With regard to the various definitions of $\mu_0, \epsilon_0, e, b, r$, the situation is quite different.

Using these constants, Maxwell's equations outside of macroscopically described matter (polarization = magnetization = 0) can be written in any system of units as

$$e\,\boldsymbol{\nabla}\cdot\boldsymbol{E} = r\,4\pi\,\rho \tag{2a}$$

$$e\,\boldsymbol{\nabla}\times\boldsymbol{E} = -\frac{b}{c}\,\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t} \tag{2b}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0 \tag{2c}$$

$$b\,\boldsymbol{\nabla}\times\boldsymbol{B} = \frac{e}{c}\,\frac{\mathrm{d}\boldsymbol{E}}{\mathrm{d}t} + r\,\frac{4\pi}{c}\,\boldsymbol{J}\ . \tag{2d}$$

$\boldsymbol{J}$ is the electrical current density. With regard to the relativistic invariance of Maxwell's equations, the time-coordinate has been written as ct. As $\boldsymbol{J}$ and ρ differ only by mechanical units, $\boldsymbol{J}$ got the general multiplier r.

With the macroscopic electrical polarization $\boldsymbol{P}$ and the macroscopic magnetization $\boldsymbol{M}$, the dielectric displacement $\boldsymbol{D}$ and the magnetizing field $\boldsymbol{H}$ (often also called magnetic field) are defined in any system of units by

$$\frac{e}{\epsilon_0}\,\boldsymbol{D} \equiv e\,\boldsymbol{E} + \frac{4\pi}{e}\,\boldsymbol{P} \tag{3a}$$

$$\mu_0 b\boldsymbol{H} \equiv b\,\boldsymbol{B} - \frac{4\pi}{b}\,\boldsymbol{M}\ . \tag{3b}$$

Then Maxwell's equations in macroscopically described matter can be written in any system of units in the form

$$\frac{e}{\epsilon_0}\,\boldsymbol{\nabla}\cdot\boldsymbol{D} = r\,4\pi\,\rho \tag{4a}$$

$$e\,\boldsymbol{\nabla}\times\boldsymbol{E} = -\frac{b}{c}\,\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t} \tag{4b}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0 \tag{4c}$$

$$\mu_0 b\,\boldsymbol{\nabla}\times\boldsymbol{H} = \frac{e}{\epsilon_0 c}\,\frac{\mathrm{d}\boldsymbol{D}}{\mathrm{d}t} + r\,\frac{4\pi}{c}\,\boldsymbol{J}\ . \tag{4d}$$

Outside of macroscopically described matter ($\boldsymbol{P} = \boldsymbol{M} = 0$) these equations turn into the equations (2). ("Outside of macroscopically described matter" does not mean "in vacuum". The vacuum instead is defined by $\boldsymbol{P} = \boldsymbol{M} = \rho = \boldsymbol{J} = 0$.)

All equations which we have used until now are valid for arbitrary systems of units. In the five systems of units, which are most commonly used in electrodynamics, the five constants $\mu_0, \epsilon_0, e, b, r$ are defined as follows:

System	μ_0	ϵ_0	b	$e \equiv \frac{1}{r}$
MKSA	$4\pi \cdot 10^{-7}\,\frac{\mathrm{N}}{\mathrm{A}^2}$	$\frac{1}{\mu_0 c^2}$	$\sqrt{\frac{4\pi}{\mu_0}}$	$\sqrt{\frac{4\pi}{\mu_0 c^2}}$
Heaviside-Lorentz	1	1	$\sqrt{4\pi}$	$\sqrt{4\pi}$
Gauß	1	1	1	1
ESU	$\frac{1}{c^2}$	1	c	1
EMU	1	$\frac{1}{c^2}$	1	$\frac{1}{c}$

(5)

The MKSA-system and the system of Heaviside and Lorentz often are called rationalized systems, because the irrational factor 4π is due to the constants canceled from Maxwell's equations. In exchange, factors 4π show up in these systems at other places, where they are absent in the other systems. Thus is is quite arbitrary which systems should be considered as rationalized or not rationalized. But it is a physically important point, that only the MKSA-system — different from all other systems listed in (5) — is defining a special unit (namely the Ampere) for the description of electromagnetic phenomena. The other systems have been invented at a time, when many physicists still assumed that they might some day succeed to embed electrodynamics into Newtons mechanics. Furthermore it is an advantage of the MKSA-system, that one notorious source of errors in the collaboration with engineers (who are using exclusively the MKSA-system, which also is called SI = systeme internationale) is eliminated upfront. The system of Heaviside and Lorentz and the system of Gauß have the remarkable advantage, that the electrical field and the magnetic field, which are the six independent components of the skew-symmetric relativistic field-strength tensor, reasonably get the same units. Certainly none of the five systems listed in (5) can be called incorrect. And certainly the assertion of some textbook authors, that students can comprehend the meaning of electrodynamics only if they are using the system of units favored in their respective books, may justly be called rubbish.

Using the constants listed in table (5), the formulas of electrodynamics can be translated relatively simple from one system of units to another. Example: The Lorentz-force, which is acting onto a charge q moving with velocity $\boldsymbol{v}$, is in the Gauß system

$$\boldsymbol{F} = q_{\text{Gauß}}\Big(\boldsymbol{E}_{\text{Gauß}} + \frac{1}{c}\,\boldsymbol{v}\times\boldsymbol{B}_{\text{Gauß}}\Big)\ . \tag{6a}$$

We want to translate this formula into the MKSA-system. As the charge q differs only by mechanical units (i. e. by an integral over position space) from the charge density ρ, the general multiplier of q is r. The general multiplier of $\boldsymbol{E}$ is e. According to (5), $e\equiv 1/r$ holds. Thus the form of the product $q\boldsymbol{E}$ is identical in all systems of units. In total one finds

$$\begin{aligned}\boldsymbol{F} &= q_{\text{MKSA}}\Big(\boldsymbol{E}_{\text{MKSA}} + \frac{1}{c}\,\frac{r_{\text{MKSA}}b_{\text{MKSA}}}{r_{\text{Gauß}}b_{\text{Gauß}}}\,\boldsymbol{v}\times\boldsymbol{B}_{\text{MKSA}}\Big)\\ &= q_{\text{MKSA}}\Big(\boldsymbol{E}_{\text{MKSA}} + \frac{1}{c}\sqrt{\frac{\mu_0c^2}{4\pi}\,\frac{4\pi}{\mu_0}}\,\boldsymbol{v}\times\boldsymbol{B}_{\text{MKSA}}\Big)\end{aligned}$$

$$\boldsymbol{F} = q_{\text{MKSA}} \Big(\boldsymbol{E}_{\text{MKSA}} + \boldsymbol{v} \times \boldsymbol{B}_{\text{MKSA}} \Big) \ . \tag{6b}$$

In the example of the Lorentz-force it was easy to find the general multiplier of the charge q by simple considerations. It's not difficult to find the general multipliers of other quantities by similar considerations. Some important examples are listed in the following table:

Quantity	general multiplicator
electric field-strength $\boldsymbol{E}$, voltage U, scalar potential Φ	e
displacement $\boldsymbol{D}$	$\frac{e}{\epsilon_0}$
polarization $\boldsymbol{P}$	$\frac{4\pi}{e}$
magnetic field-strength (induction) $\boldsymbol{B}$, vector potential $\boldsymbol{A}$	b
magnetizing field $\boldsymbol{H}$	$\mu_0 b$
magnetization $\boldsymbol{M}$	$\frac{4\pi}{b}$
charge density ρ, charge q, current I, current density $\boldsymbol{J}$	$\frac{1}{e}$
resistance R, inductance L	$\frac{e}{r} = e^2$
capacity C, conductivity σ	$\frac{1}{e^2}$

(7)