

# The three types of entropy

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Under the notion entropy, which is of fundamental importance for thermodynamics and statistics, actually three concepts with significant differences are subsumed. Use of the identical name for different types of entropy resulted into appreciable confusion in the literature. In this article the basic conceptual features of the three approaches are reviewed, and their differences are pointed out.

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## 1. Introduction

The concept of entropy is a cornerstone in the foundations of thermodynamics and statistics. Actually three variants of entropy, with significant differences, are subsumed under this name, and all three are in use by today. The first type of entropy has been conceived by Clausius [1, 2] in the eighteen-fifties. About twenty years later, Gibbs [3] introduced a basic change into the definition of this notion. Probably Gibbs was not aware that he was significantly deviating from Clausius' concept; in any case the split of concepts was not appropriately discussed in the literature, thus the confusion started.

When Shannon analyzed the information content of signals by end of the nineteen-forties, and conceived a function for the quantitative description of the “choice” or “uncertainty” of information,

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famously v. Neumann suggested to name this function entropy, “because anyway nobody knows what entropy is.” Instead of clarifying the issue and removing the confusion, which v. Neumann ironically — but correctly — stated in this anecdote, Shannon increased it by adding a third concept of entropy, again with significant differences versus the older definitions.

Of course there is much common ground and overlap between the three entropy concepts; otherwise they would have never made their way into the textbooks under a unique name. But the differences are substantial.

The three different types of entropy can be classified with regard to the objects, to which the entropy is assigned:

	object type	interpretation of the object
(ta)-entropy	tangible	analog
(td)-entropy	tangible	digital
(id)-entropy	information	digital

Tab. 1: The three types of entropy

Thus in this nomenclature the characters in brackets are indicating the classification criteria. The objects are often called “systems”. Tangible systems are aggregates of matter, like e. g. a solid, or some gas enclosed in a vessel. Information, on the other hand, may be a signal, or a stream of signals, arriving from a telecommunication line, or it may be carried by a sequence of characters written in a book. Not the voltage of the communication line, and not the ink marks on the paper, but the meaning which the reader assigns to the voltage or to the characters are the information. Information is residing on an ontological meta-level, different from the level of tangible objects.

To understand what is meant by analog versus digital interpretation of an object, look at this example:

$$A \quad A \quad A \quad \mathcal{A} \quad \sphericalangle \quad \triangleright \quad \mathfrak{A} \quad (1a)$$

It's easy to see that these seven connected<sup>2</sup> graphs all are different. To document the graphs and their differences, we could for example overlay the graphs with a fine coordinate grid, and compile in a list which squares of the grid are black, and which are white. This would be an analog interpretation of the graphs.

In contrast, in a digital interpretation we may arbitrarily define, that each of these seven graphs is nothing but a representation of the character 0100 0001 of the 8-bit ASCII code, i. e. the capital Latin letter A. In an alternative digital interpretation we may arbitrarily define that

$$\begin{aligned} A \text{ codes for } e, \quad A \text{ codes for } n, \quad A \text{ codes for } t, \quad \mathcal{A} \text{ codes for } r, \\ \sphericalangle \text{ codes for } o, \quad \triangleright \text{ codes for } p, \quad \mathfrak{A} \text{ codes for } y, \end{aligned} \quad (1b)$$

and thereby interpret the seven graphs as

$$(1a) = \text{entropy} . \quad (1c)$$

The graphs (1a) are sufficient for an analog interpretation (the coordinates list of the pixels). But you can not arrive at the digital interpretation (1c), unless you have both the graphs (1a) and the rules (1b) available. Generally speaking: An analog interpretation of an object is fully determined by the object itself, while any digital interpretation of the same object is only determined by the combination of that object and some interpretative rules.

There exists no (ia)-entropy, because that combination would be a contradiction in itself. Signals are pieces of transmitted information. Without digital interpretation, what is received from

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<sup>2</sup> A graph is connected, if from any black point of the graph in a paper print any other black point of this graph can be reached without crossing a white gap.

a telecommunication line would be a “varying voltage”, but not a “signal”. Information rests in any case on the digital interpretation of some object.

As (id)-entropy is the only type of entropy with an (i), one might be inclined to abbreviate the nomenclature to (i)-entropy, the more so as (id)-entropy is often called information-entropy in the literature. I will not do that in this article, however, for two reasons: First, I think it is an advantage to have the structure as outlined in table 1 at any time visible. Second, I will use in the sequel “(d)-entropy” as an abbreviation for “(td)- and (id)-entropy”, and “(t)-entropy” as an abbreviation for “(ta)- and (td)-entropy”.

The (t) of (t)-entropy, standing for “tangible”, may alternatively be read as “thermodynamic”, because both (ta)-entropy and (td)-entropy have been conceived within the framework of thermodynamics. The interpretation of (t) as “tangible” has the advantage, however, to emphasize the essential physical difference between the concepts of the two types of (t)-entropy and the concept of (id)-entropy: (ta)-entropy and (td)-entropy are assigned to tangible objects, while (id)-entropy is assigned to information (which may be information about the state of a tangible object).

Looking at the references, the reader might get the impression that this is an article about the history of physics, given the many citations from the 19<sup>th</sup> and 20<sup>th</sup> century. That impression would be completely wrong, however. Instead I aim — for the benefit of scientists (and students!) in the 21<sup>st</sup> century — to reduce as good as possible the confusion about the meaning of entropy. This aim, I think, can best be reached by tracing the splits between the various concepts of entropy in historical perspective.

I will start in section 2 with the introduction of (ta)-entropy due to Clausius [1, 2] in the framework of phenomenological thermodynamics, and the (not fully successful) extension of this notion to

statistical thermodynamics due to Boltzmann [4]. The step to (td)-entropy, by which Gibbs [3] removed the problems of Boltzmann’s attempt, is discussed in section 3. In section 4 Shannon’s [5] (id)-entropy is treated, which extended the notion of entropy beyond the range of thermodynamics. (Actually Gibbs [6] had already initialized a good part of this extension.) Eventually in section 5 the findings and conclusions of this article are briefly compiled.

## 2. (ta)-entropy

Thermodynamics are characterized by the delicate balance between a symmetry and an asymmetry. The symmetry is the homogeneity of time, which according to Noether’s theorem implies the conservation of energy.

The first law of thermodynamics:

$$dU = \delta Q + dW \quad (2)$$

Here  $U$  is the inner energy of the considered system.  $Q$  is the heat, and  $W$  — usually read as “work” — are all other forms of energy except of heat, which the system exchanges<sup>3</sup> with the environment.

While the first law imposes no restrictions onto transformations between the various forms of energy, experience shows that other forms of energy can be 100 % transformed into heat, but heat can not be 100 % transformed into other forms of energy. In the words of Kelvin [7]:

“It is impossible, by means of inanimate material agency, to derive mechanical effect from any portion of matter by cooling it below the temperature of the coldest of the surrounding objects.”

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<sup>3</sup> We apply the convention that energy, which is fed from outside into the system, gets a positive sign ( $Q > 0$ ,  $W > 0$ ), while energy, which the system feeds towards the environment, gets a negative sign ( $Q < 0$ ,  $W < 0$ ).

This implies an inhomogeneity of time: While the total energy content of a closed system is constant, the share of heat may increase with increasing time, but it will never decrease. The apparent contradiction is only an artifact, however, caused by the inadequacy of the human perspective. Maxwell [8, pp. 153, 154] explains:

“If heat in a body consists in a motion of its parts, and if we were able to distinguish these parts, and to guide and control their motions by any kind of mechanism, then by arranging our apparatus so as to lay hold of every moving part of the body, we could, by a suitable train of mechanism, transfer the energy of the moving parts of the heated body to any other body in the form of ordinary motion. The heated body would thus be rendered perfectly cold, and all its thermal energy would be converted into the visible motion of some other body.

Now this supposition involves a direct contradiction to the second law of thermodynamics, but is consistent with the first law. The second law is therefore equivalent to a denial of our power to perform the operation just described, either by a train of mechanism, or by any other method yet discovered. Hence, if the heat of a body consists in the motion of its parts, the separate parts which move must be so small or so impalpable that we cannot in any way lay hold of them to stop them.”

Thus, if we were able to observe and control kinetic energy on the molecular scale as easily as on the macroscopic scale, then no human being would ever have invented the notion “heat”. Instead we would observe that kinetic energy is merely shuffled between various degrees of freedom of matter. A fortiori no human being would ever have stated the second law of thermodynamics. Instead the homogeneity of time would be reflected undisturbed in all

theories of classical physics.<sup>4</sup>

Clausius searched for a precise quantitative description of the asymmetry in the transformations between heat and other forms of energy. Due to analysis of the cyclic Carnot process [9], he detected that not the heat  $Q$ , but the quotient heat divided by temperature  $Q/T$  is the crucial quantity. He defined [1,2] the (ta)-entropy  $S$  due it's infinitesimal change:

$$dS = \left. \frac{\delta Q}{T} \right|_{\text{rev}} = \begin{array}{l} \text{(ta)-entropy change of a system,} \\ \text{which reversibly exchanges}^3 \text{ the} \\ \text{heat } \delta Q \text{ with a bath of tempera-} \\ \text{ture } T \end{array} \quad (3)$$

Clausius used in his writings the capital Latin letter  $S$  for (ta)-entropy, Gibbs used the lowercase Greek letter  $\eta = \text{eta}$  for (td)-entropy, and Shannon used the capital Greek letter  $H = \text{Eta}$  for (id)-entropy. I will follow in this article these conventions.

Clausius proved that by means of the definition (3) the asymmetry of transformations between heat and other forms of energy can be stated as

the second law of thermodynamics:

The (ta)-entropy of a system can not decrease, unless that decrease is compensated or over-compensated by an increase of (ta)-entropy of the environment. (4)

$$dS_{\text{system}} + dS_{\text{bath}} \begin{cases} = 0 & \text{for reversible changes} \\ > 0 & \text{for irreversible changes} \end{cases}$$

Like the first law, the second law is a law of nature. It can not be derived from other, more basic laws. Instead it was found by guessing, with the guessing being guided by experimental experience.

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<sup>4</sup> I will not consider in this article the intricate complications with respect to the notion "time", which came in due to quantum theory.

The seven graphs (1a) were in one digital interpretation identified as seven times the letter A, with the graphical differences being irrelevant. In the digital interpretation (1b), the same graphs coded for the word “entropy”. Countless further digital interpretations could be invented. In contrast, the analog documentation, i. e. the coordinate list of black and white patches, does not depend on additional interpretation rules. Likewise the (ta)-entropy change

$$\Delta S = S_B - S_A = \int_A^B dS \stackrel{(3)}{=} \int_A^B \frac{\delta Q}{T} \quad (5)$$

reversible

does not depend on arbitrarily defined interpretation rules. By tracing the temperature  $T$  of the system, and its heat exchange  $Q$  with the environment, different physicists will measure identical values, if they analyze the (ta)-entropy changes in course of identical processes. It’s a characteristic feature of an analog interpreted property, that it can be objectively measured. We will see that the unique, objective measurability is a particular feature, by which (ta)-entropy differs from both (td)-entropy and (id)-entropy.

Here a slight complication must be mentioned: Clausius emphasized [1, § 16] the difference

$$d\Omega = \frac{\delta Q}{T} \Big|_{\text{irreversible}} \neq \frac{\delta Q}{T} \Big|_{\text{reversible}} \stackrel{(3)}{=} dS \quad (6)$$

between the entropy  $S$  and the quantity  $\Omega$ , which he named “not compensated transformation” (in 19<sup>th</sup> century German: „uncompensirte Verwandlung“). In an irreversible process,  $\delta Q$  may be completely different from  $T dS$ . In particular,  $\delta Q$  may be zero, even if  $T dS \neq 0$ . The irreversible expansion of an ideal gas is the paradigmatic example for that case.

Therefore the change of (ta)-entropy in course of an irreversible process can be measured only indirectly. As



$$\oint_{\text{reversible}} dS = \oint_{\text{reversible}} \frac{\delta Q}{T} \stackrel{(3)}{=} 0 \quad (7)$$

holds for a cyclic process, the entropy difference  $\Delta S_{BA} = S_B - S_A$  does not depend on the particular path along which the system is reversibly transformed from state  $A$  to state  $B$ . Instead  $S_A$  and  $S_B$  are well-defined state functions, even though their absolute values have not been defined. This implies that the entropy change  $\Delta S_{BA}$  is identical for reversible and irreversible transitions from state  $A$  to state  $B$ , notwithstanding  $\int_A^B \delta Q/T$  being different for reversible and irreversible transitions. Therefore the entropy change  $\Delta S_{BA}$  of an irreversible transition from state  $A$  to state  $B$  can indirectly be measured either by measuring  $\int \delta Q/T$  in course of a reversible transition  $A \rightarrow B$  or in course of a reversible transition  $B \rightarrow A$ :

$$\Delta S_{BA,\text{irreversible}} = \Delta S_{BA,\text{reversible}} = -\Delta S_{AB,\text{reversible}} \quad (8)$$

Thus there is no doubt how changes  $\Delta S$  of (ta)-entropy have to be measured.

The rule for the measurement of changes of (ta)-entropy is uniquely specified by the definition (3). Consequently, if two physicists measure a change of (ta)-entropy, they will in any case agree on the result. (9)

We will see in section 4 that changes of (id)-entropy are not objective events, but describe changes of knowledge of an observer. Hence it is worthwhile to make the following statement with regard to (ta)-entropy:

If in course of a process the (ta)-entropy of a system changes, then this change is an objective event, no matter whether or not a physicist takes note of it. (10)  
 If two physicists take note of it, they will observe the identical value of (ta)-entropy change.

The difference between (ta)-entropy and (td)-entropy can most clearly be demonstrated by the example of expansion and mixture of ideal gases. We will evaluate three different processes:

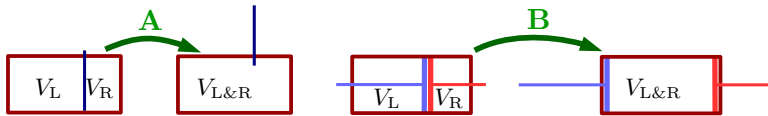


Fig. 1: Expansion and mixture of ideal gases

- ① The left compartment (volume  $V_L$ ) of the vessel sketched in fig. 1 A is filled by an ideal gas. The right compartment (volume  $V_R$ ) is filled by an ideal gas of different kind. By pulling out the partition, both gases are irreversibly expanded and mixed to the volume  $V_{L\&R}$ .
- ② The left compartment of the vessel is filled by an ideal gas, and the right compartment is filled by an ideal gas of different kind. Both gases are reversibly expanded and mixed to the volume  $V_{L\&R}$  due to pulling out semi-permeable pistons as displayed in fig. 1 B: The left piston is permeable for the gas from the left compartment, but impermeable for the gas from the right compartment, and vice versa. It's of no relevance whether we actually have such semi-permeable pistons available. The fact that such semi-permeable pistons are conceivable (i. e. that such pistons are compatible with all known laws of nature) justifies their application in the theory of thermodynamics.
- ③ Both compartments of the vessel sketched in fig. 1 A are filled by ideal gases of identical kind. By pulling out the partition, the both gases are mixed and expanded to the volume  $V_{L\&R}$ . While this is a reversible process within the continuum model of matter, it's an intricate question (“Gibbs’ paradox”) whether or not this process is reversible within the molecular-kinetic model of matter. Note that this process can not be performed

with semi-permeable pistons, because such pistons can discern atoms and molecules of different type, but they can not discern atoms and molecules of identical type, which merely differ by their provenience from the left or the right compartment.

In any case we stipulate that all gases have the same pressure at start of the processes. Furthermore the vessel is embedded in a thermal bath of temperature  $T$ . This secures that all gases have temperature  $T$  at any time. First we discuss the processes in the continuum-model of phenomenological thermodynamics.

Process ① (irreversible mixture of two different gases, fig. 1 A): Being ideal gases, each gas behaves as if the other gas didn't exist. The gases do no work  $W$  upon expansion, and the temperature of the gases does not change, as experimentally confirmed by Gay-Lussac [10]. Consequently there is no exchange of heat  $Q$  with the environment:

$$W = 0 \quad , \quad Q = 0 \quad (11)$$

Process ② (reversible mixture of two different gases, fig. 1 B): We exploit the equation

$$\frac{pV}{T} = C = \text{constant} > 0 \quad (12)$$

of ideal gases, to compute the work<sup>3</sup>

$$-W = \int_{V_L}^{V_{L\&R}} dV p + \int_{V_R}^{V_{L\&R}} dV p \stackrel{(12)}{=} C_L T \ln \frac{V_{L\&R}}{V_L} + C_R T \ln \frac{V_{L\&R}}{V_R} \quad , \quad (13)$$

which the gases do against the pistons. Thereby the gases would cool down, unless the heat

$$Q = -W = C_L T \ln \frac{V_{L\&R}}{V_L} + C_R T \ln \frac{V_{L\&R}}{V_R} \quad (14)$$

would be extracted from the bath. The (ta)-entropy change

$$\Delta S \stackrel{(3)}{=} \left. \frac{Q}{T} \right|_{\text{rev}} \stackrel{(14)}{=} C_L \ln \frac{V_{L\&R}}{V_L} + C_R \ln \frac{V_{L\&R}}{V_R}, \quad (15)$$

which is computed — according to Clausius' definition (3) — from the reversible process ②, is valid as well for the irreversible process ①. Note that in this case

$$\left. \frac{Q}{T} \right|_{\text{irreversible}} \stackrel{(11)}{=} 0 \stackrel{(15)}{\neq} \Delta S > 0, \quad (16)$$

confirming Clausius' assertion (6).

Process ③ (mixture of identical gases): As the constants  $C_L$  and  $C_R$  are independent of the chemical type of the gases, and consequently  $\Delta S = (15)$  as well is independent of the chemical type of the gases, one might naively guess that we get again  $\Delta S = (15) > 0$ . But this is not correct. (15) has been derived from a process, in which semi-permeable pistons were used. The mixture with the same gas in both compartments, however, can not be accomplished by semipermeable pistons. But the reversible mixture can easily be accomplished with the setup fig. 1 A by simply pulling-out the partition. To reverse the mixture, we simply shift-in the partition again. According to the ideal-gas equation, we have before the mixture

$$\frac{pV_R}{T} \stackrel{(12)}{=} C_R, \quad \frac{pV_L}{T} \stackrel{(12)}{=} C_L. \quad (17a)$$

After the partition has been pulled out, this becomes

$$\frac{pV_{L\&R}}{T} = \frac{p(V_R + V_L)}{T} \stackrel{(12)}{=} C_L + C_R. \quad (17b)$$

Shifting the partition in again, we return to (17a). Shifting out and in the partition does not change any parameter which impacts  $\Delta S$ . No work needs to be done to shift the partition in or out, nor is any heat exchanged with the environment. Hence we find the change of (ta)-entropy

$$\Delta S = 0 \tag{18}$$

upon mixture of ideal gases of same type, same pressure, and same temperature.

Note that all changes of (ta)-entropy upon expansion and/or mixture of gases are (in case of irreversible processes only indirectly) measurable, objective quantities. We will see that this is a particular feature of (ta)-entropy. In contrast, the values of changes of (td)-entropy depend on additional interpretative rules, and changes of (id)-entropy are not at all objectively measurable.

Now we analyze the processes ①, ②, ③ a second time, this time using the statistical methods of the molecular-kinetic model of gases. In this model, the constant  $C$  within the equation (12) of ideal gases is replaced by the product  $Nk$  of the number  $N$  of atoms or molecules, which constitute the gas, and the Boltzmann constant  $k$ .

The molecular-kinetic model assumes, that the observable macrostate of the gas, which is characterized by observable parameters like the pressure  $p$ , the temperature  $T$ , and the volume  $V$ , is caused by it's not directly observable micro-state  $\prod_{j=1}^N x_j y_j z_j p_{xj} p_{yj} p_{zj}$ , i. e. the product of the phase coordinates of all it's  $N$  constituent particles.

The microstate  $\prod_{j=1}^N x_j y_j z_j p_{xj} p_{yj} p_{zj}$  can not be uniquely concluded from the observed macrostate  $\{p, T, V\}$ . Instead the macrostate usually is compatible with a huge number of different microstates. For the entropy difference between two macrostates  $B$  and  $A$  of the gas, Boltzmann [4] postulated

$$\Delta S_{BA} = S_B - S_A = k \ln \frac{L_B}{L_A} = k \ln \frac{\Phi_B}{\Phi_A}, \quad (19)$$

with  $L_B/L_A$  being the quotient of the number of different microstates which are compatible with the macrostates  $B$  and  $A$ , respectively. But what is the value of  $L_B/L_A$ ? Boltzmann [4] postulated  $L_B/L_A = \Phi_B/\Phi_A$ , with

$$\Phi = \prod_{j=1}^N \iiint_{\text{accessible}} dx_j dy_j dz_j dp_{xj} dp_{yj} dp_{zj} \quad (20)$$

being the phase space accessible to the gas. ‘‘Accessible’’ means, that the integrals over the positions of the  $N$  atoms are only limited by the boundaries of the vessel, and the integrals over their momenta are only limited by the total inner energy of the gas.

Now we apply Boltzmann’s postulate (19) to the processes ① and ② (irreversible and reversible mixture of two different gases, fig. 1 A and B): Due to expansion and mixture of the two gases, the accessible phase space increases:

$$\Phi_{L\&R} \stackrel{(20)}{=} \left( \frac{V_{L\&R}}{V_L} \right)^{N_L} \Phi_L \cdot \left( \frac{V_{L\&R}}{V_R} \right)^{N_R} \Phi_R \quad (21)$$

Consequently the expansion and mixture results into this increase of (ta)-entropy:

$$\Delta S \stackrel{(19)}{=} k \ln \frac{\Phi_{L\&R}}{\Phi_L \Phi_R} \stackrel{(21)}{=} N_L k \ln \frac{V_{L\&R}}{V_L} + N_R k \ln \frac{V_{L\&R}}{V_R} = (15) \quad (22)$$

Thus with Boltzmann’s postulate (19), the change (15) of (ta)-entropy as computed by the methods of phenomenological thermodynamics is indeed reproduced by the methods of statistical thermodynamics.

Process ③ (mixture of identical gases): As we stick to the condition of same gas pressure  $p$  and same temperature  $T$  in both

compartments, we get from the ideal gas equation (17a) with  $C = Nk$

$$\frac{V_L}{V_R} = \frac{N_L}{N_R} \quad (23a)$$

$$N_L + N_R = N \quad , \quad V_L + V_R = V_{L\&R} . \quad (23b)$$

From (20) we see that the phase space accessible to the gas from the left and right compartments increases upon removal of the partition by the factors

$$\left(\frac{V_{L\&R}}{V_L}\right)^{N_L} \stackrel{(23)}{=} \left(\frac{N}{N_L}\right)^{N_L} \quad (24a)$$

$$\left(\frac{V_{L\&R}}{V_R}\right)^{N_R} \stackrel{(23)}{=} \left(\frac{N}{N_R}\right)^{N_R} . \quad (24b)$$

Thus after the mixture the accessible phase space is

$$\Phi_{L\&R} \stackrel{(20),(24)}{=} \left(\frac{N}{N_L}\right)^{N_L} \Phi_L \cdot \left(\frac{N}{N_R}\right)^{N_R} \Phi_R , \quad (25)$$

and the change of (ta)-entropy upon mixture of identical gases becomes

$$\Delta S \stackrel{(19)}{=} k \ln \left(\frac{\Phi_{L\&R}}{\Phi_L \Phi_R}\right) = k \ln \left(\frac{N^N}{N_L^{N_L} N_R^{N_R}}\right) \neq (18) . \quad (26)$$

This result is different from the result  $\Delta S \stackrel{(18)}{=} 0$  of phenomenological thermodynamics! The mismatch (18)  $\neq$  (26) became well-known under the name ‘‘Gibbs’ paradox’’.

A solution (?) for Gibbs’ paradox came with quantum theory: States of a gas, which differ merely by permutations of atoms or molecules of same type, are one single state according to quantum theory. Consequently Boltzmann’s formula (19) must be amended to

$$\Delta S_{BA} = k \ln \frac{\Phi_B / N_B!}{\Phi_A / N_A!} . \quad (27)$$

The result (22) for expansion and mixture of different gases is not affected by the amendment (27), because the additional factors  $N_{R!}$  and  $N_{L!}$  cancel out. But in case of mixture of identical gases we get with Stirling's formula  $\ln N! = \ln N^N + \mathcal{O}(N)$  instead of (26)

$$\Delta S = k \ln \left( \frac{N^N N_L! N_R!}{N! N_L^{N_L} N_R^{N_R}} \right) \stackrel{N \ln N \gg N}{\approx} k \ln 1 = 0 . \quad (28)$$

Some physicists are satisfied by this “solution” of Gibb's paradox. But we should be concerned because the problem remains in case of small  $N$ . Gibbs proposed a *much* better solution.

### 3. (td)-entropy

The difference between the entropy change  $\Delta S \neq 0$  upon mixture of different gases, and  $\Delta S = 0$  upon mixture of identical gases seemed perfectly natural and not at all paradox to Gibbs himself, half a century before the advent of quantum theory. Gibbs was the first to understand that within statistical thermodynamics only (td)-entropy, but not (ta)-entropy, can be consistently defined:<sup>5</sup>

According to Clausius' definition (3), the change of (ta)-entropy upon transition from state A to state B is

$$\Delta S_{BA} \stackrel{(3)}{=} \int_A^B \frac{\delta Q}{T} . \quad (29a)$$

reversible

Boltzmann had replaced this by

<sup>5</sup> I (gg) became aware of this fact due to a beautiful article by Jaynes [11].



$$\Delta S_{BA} \stackrel{(19)}{=} k \ln \frac{L_B}{L_A} = k \ln \frac{\Phi_B}{\Phi_A}, \quad (29b)$$

which later had been amended to

$$\Delta S_{BA} = k \ln \frac{L_B}{L_A} \stackrel{(27)}{=} k \ln \frac{\Phi_B / N_B!}{\Phi_A / N_A!}, \quad (29c)$$

with  $L$  being the number of different microstates, which are compatible with the observed macrostate. The essential point is not the question whether  $L \sim \Phi$  or  $L \sim \Phi/N!$  or  $L \sim$  whatever is correct. Instead the essential point — which Gibbs noticed but Boltzmann had overlooked — is, that  $\delta Q/T$  in (29a) is an objectively and uniquely measurable quantity, i. e. an analog interpreted quantity. In contrast, the number  $L$  is *not* an analog, objectively measurable quantity.  $L$  is not an analog quantity, because the “state”, which is assigned to a system, is not an analog property of the system. In fact the state is assigned to the system according to arbitrarily defined, digital criteria.

To make this concrete, let’s assume that we used Argon as an (almost) ideal gas in the mixing experiment of two identical gases. Now consider this scenario: When we filled Argon into the vessel before start of the gas-mixing experiment, the Argon bottle was almost empty when the left compartment was filled. Therefore we took the Argon for the right compartment from a new bottle. After the experiment, looking again onto the labels of the bottles, we suddenly become aware that the first bottle was filled with high-purity  $^{36}\text{Ar}$ , while the second bottle is filled with high-purity  $^{40}\text{Ar}$ . Thus we had at start of the experiment pure  $^{36}\text{Ar}$  in the left compartment and pure  $^{40}\text{Ar}$  in the right compartment. After removal and re-insertion of the partition, we now have in both compartments a mixture of  $^{36}\text{Ar}$  and  $^{40}\text{Ar}$ .

Does this mean that our previous result (28) is wrong? Gibbs understood that this question can not be reasonably answered by

a simple yes or no. Instead the answer depends on an *arbitrary definition*. Like in one digital interpretation of the seven graphs (1a) the objective differences are irrelevant, and all seven graphs code for the letter A, while in another digital interpretation the differences are important and have the meaning stated in (1b), we can (and must!) as well decide for an arbitrary definition when assigning a “state” to the gas:

In the gas mixing experiments we can control parameters like  
 the pressure  $p$             the temperature  $T$     the volume  $V$   
 the chemical type  $\mu$     the isotope type  $\zeta$   
 of the gas. It is our arbitrary decision, whether we want to define more or less or other parameters as “relevant”. If we decide for  $\{p, T, V, \mu\}$ , then the state of the Argon gas after mixture and reinsertion of the partition is *identical by definition* to it’s state before the experiment, and the change of (td)-entropy upon mixture is

$$\Delta\eta = 0 . \quad (30a)$$

If we declare  $\{p, T, V, \mu, \zeta\}$  as relevant criteria, however, then we assign to the gas different states before and after the mixture. We may imagine that we have semi-permeable pistons available, which are permeable only for  $^{36}\text{Ar}$  or permeable only for  $^{40}\text{Ar}$ . When the gas is reversibly expanded by means of these pistons, it does the work<sup>3</sup>

$$-W \stackrel{(13)}{=} N_{\text{L}}kT \ln \frac{V_{\text{L\&R}}}{V_{\text{L}}} + N_{\text{R}}kT \ln \frac{V_{\text{L\&R}}}{V_{\text{R}}} , \quad (30b)$$

and thereby absorbs from the bath the heat<sup>3</sup>

$$Q = -W . \quad (30c)$$

Thus the (td)-entropy of the gas increases by

$$\Delta\eta = \frac{Q}{T} = N_{\text{L}}k \ln \frac{V_{\text{L\&R}}}{V_{\text{L}}} + N_{\text{R}}k \ln \frac{V_{\text{L\&R}}}{V_{\text{R}}} . \quad (30d)$$

Note that the heat  $Q = (30c)$  is absorbed only if the gases are expanded by the semi-permeable pistons. If we expand the gases by pulling out the partition, then they do not absorb heat upon mixture (remember the experiment of Gay-Lussac). Thus, if we hadn't read the labels of the Ar bottles after the experiment, we would have never noticed that different isotopes of Argon have been used, even if we had controlled the process by sensitive calorimeters.

By an arbitrary digital definition, we decide for the state function  $\{p, T, V, \mu\}$ , or  $\{p, T, V, \mu, \zeta\}$ , or whatever. As the state function determines the number  $L$  of microstates which are compatible with that state, and consequently the entropy (29b) resp. (29c), this means that we actually have switched from (ta)-entropy to (td)-entropy.

One might object: Why should we arbitrarily ignore a parameter like  $\zeta$  (the isotope type)? Should we not in any case assign to any system that unique state function, which is complete and objectively correct? The important answer is:

*That “unique, complete, objectively correct”  
state function does not exist!*

That is exactly, what we can learn from the  $^{36}\text{Ar} / ^{40}\text{Ar}$  isotope example. Gibbs knew nothing about isotopes. Hence he was during all his lifetime unable to write down an objectively complete and correct state function of gases or any other type of matter. And we? Can we be sure that future scientist will not detect properties of matter which are unknown to us? Gibbs envisaged that a chemical element  $X$  might — unknown to us — actually consist of two different types of atoms  $X_a$  and  $X_b$ , and that  $X_a$  would react with a yet unknown element  $Y$ , but be inert against a yet unknown element  $Z$ , while  $X_b$  would react with  $Z$ , but be inert against  $Y$ . And he envisaged that objectively different types of gases might be so similar, that only future scientists will be able

to discern them due to advanced technology. These are Gibbs' considerations [3, part I, pp. 228–229]:

“Now we may without violence to the general laws of gases which are embodied in our equations suppose other gases to exist than such as actually do exist, and there does not appear to be any limit to the resemblance which there might be between two such kinds of gas. But the increase of entropy due to the mixing of given volumes of the gases at a given temperature and pressure would be independent of the degree of similarity or dissimilarity between them. We might also imagine the case of two gases which should be absolutely identical in all the properties (sensible and molecular) which come into play while they exist as gases either pure or mixed with each other, but which should differ in respect to the attractions between their atoms and the atoms of some other substances, and therefore in their tendency to combine with such substances. In the mixture of such gases by diffusion an increase of entropy would take place, although the process of mixture, dynamically considered, might be absolutely identical in its minutest details (even with respect to the precise path of each atom) with processes which might take place without any increase of entropy.”

Gibbs understood that we run into absurdities, if we cling to (ta)-entropy: We can never be sure that two states of a system, which seem identical to us, really are “objectively” identical. The notion of entropy becomes applicable only if we at some point of our considerations declare two states of a system *identical by definition*, whether or not objective differences between these two states may be known by today or might be detected in future. This of course means: The notion of entropy becomes applicable only if we switch from (ta)-entropy to (td)-entropy.

Note that these considerations only apply in the range of statistical thermodynamics. Clausius' definition (29a) of (ta)-entropy rests on the uniquely measurable (analog) quantities  $Q$  and  $T$ , and there is no problem with these quantities in the range of phenomenological thermodynamics. But in statistical thermodynamics, the definition (29b) resp. (29c) of entropy rests on the number  $L$  of different microstates which are compatible with the macrostate of the system. When assigning a state to the observed system, we are facing problems, because there may exist many parameters, either unknown to today physics or so hard to measure that we arbitrarily decided to ignore them, which may be indispensable for a "complete" description of the state.

Of course Boltzmann had declared the phase coordinates of the single molecules as irrelevant parameters, and thereby implicitly made a first, important step towards (td)-entropy. But only Gibbs understood that there are many further parameters, which unavoidably must be declared irrelevant, to make the concept of entropy applicable within statistical thermodynamics.

It is impossible to assign a state to a system without selecting arbitrarily a finite set of parameters, declare them as the only relevant parameters, and base the definition of the state onto this incomplete selection. Gibbs comments succinctly: "It is to states of systems thus incompletely defined that the problems of thermodynamics relate." [3, part I, p. 228] Consequently — to stay consistent — we must switch from (ta)-entropy to (td)-entropy as soon as the definition of entropy rests on the definition of "states".

This is an important characteristic of (id)- and (td)-entropy: Whenever we assign a value of (d)-entropy to a system, we confine our attention to a finite, arbitrarily chosen number of it's properties which we call the relevant properties, and ignore all it's other properties:

The criteria, according to which (d)-entropy is computed, are *defined arbitrarily* by the physicist. Consequently different observers may correctly assign completely different (d)-entropies to the identical system at the same point of time, if they defined different relevant criteria. (31)

“Gibbs’ paradox” of (ta)-entropy has been resolved (by taking into account the permutability of molecules of same type) only in the limit  $N \ln N \gg N$ , see (28). For small particle numbers, the problem remains until today. In the framework of (td)-entropy, however, we are free to fix by arbitrary definitions whether objectively different states are to be considered equal, or not. That means that we are free to replace the (ta)-entropy formula

$$\Delta S_{BA} \stackrel{(27)}{=} k \ln \frac{\Phi_B / N_B!}{\Phi_A / N_A!} \quad (32a)$$

by the (td)-entropy formula

$$\Delta \eta_{BA} = k \ln \frac{\Phi_B / N_B^{N_B}}{\Phi_A / N_A^{N_A}}, \quad (32b)$$

which gives results in accord with phenomenological thermodynamics for all discussed examples of gas expansion and gas mixture, including gases of low particle number.

Gibbs appends to these considerations a somewhat cryptic remark [3, part I, p. 229]:

“Again, when such gases have been mixed, there is no more impossibility of the separation of the two kinds of molecules in virtue of their ordinary motions in the gaseous mass without any especial external influence, than there is of the separation of a homogeneous gas into the same two parts into which it has once been divided,

after these have once been mixed. In other words, the impossibility of an uncompensated decrease of entropy seems to be reduced to improbability.”

Jaynes [11] undertook “the decipherment of this into plain English”, and proposed this translation:

“It might happen without any intervention on our part that in the course of their motion the molecules which came from  $V_1$  all return to it at some later time. Such an event is not impossible; we consider it only improbable. Now a separation that Nature can accomplish already in the case of identical molecules, she can surely accomplish at least as easily for unlike ones. The spontaneous separation of mixed unlike gases is just as possible as that of like ones. In other words, the impossibility of an uncompensated decrease of entropy seems to be reduced to improbability.”

Note that this mutation of the second law of thermodynamics from a deterministic law to a merely probabilistic law did *not* come in due to Gibbs’ move from (ta)-entropy to (td)-entropy. Instead it came in due to the move from the continuum theory of matter, onto which Clausius’ original definition of entropy is based, to the molecular-kinetic theory of matter in the work of Boltzmann and Gibbs.

For the mixture of two different ideal gases we computed above the entropy increase

$$\Delta\eta \stackrel{(30d)}{=} N_L k \ln \frac{V_{L\&R}}{V_L} + N_R k \ln \frac{V_{L\&R}}{V_R} . \quad (33)$$

The probability  $P$  “of the separation of the two kinds of molecules in virtue of their ordinary motions in the gaseous mass without any especial external influence” is

$$P = \left( \frac{V_L}{V_{L\&R}} \right)^{N_L} \cdot \left( \frac{V_R}{V_{L\&R}} \right)^{N_R} \stackrel{V_L \approx V_R}{\approx} \left( \frac{1}{2} \right)^{N_L + N_R}. \quad (34)$$

This complete de-mixing with uncompensated entropy decrease  $\Delta\eta = -(\text{33})$  would of course be the most extreme case. A much less pronounced partial de-mixing would already be observable. Considering (34), we get — instead of the deterministic second law (4) of phenomenological thermodynamics — in the statistical theory this probabilistic form of

the second law of thermodynamics:

The probability  $P$  of a significant decrease of (td)-entropy, which is not compensated or over-compensated by an increase of (td)-entropy of the environment of the considered system, is by order of magnitude

$$P \approx 2^{-N},$$

with  $N$  being the number of constituent particles of the system. (35)

Thus a realistic chance to observe an uncompensated decrease of (td)-entropy exists only if the particle number  $N$  of the observed system is *very* small in comparison with Avogadro's number  $6 \cdot 10^{23}$ . Indeed the first experimental confirmations of uncompensated (td)-entropy decrease have been reported [12, 13] for the heat production (i. e. entropy production) due to viscous friction, when particles of  $6 \mu\text{m}$  diameter were dragged through an aqueous solvent.

Besides the move from (ta)-entropy to (td)-entropy, Gibbs introduced [6, chap. IV] this further improvement: Boltzmann's entropy formula

$$S \stackrel{(19)}{=} k \ln L \quad (36)$$

assumes the same probability



$$p_j = \frac{1}{L} \quad \text{for } j = 1, 2, 3, \dots, L \quad (37a)$$

for all  $L$  microstates, which are compatible with the observed macrostate. This assumption gives good results in case of an ideal gas, but it is a poor assumption when interactions between the constituent particles of the system are not negligible. To adapt (36) to cases with different  $p_j$ , we write  $L$  — still assuming (37a)

— as

$$L = L^{L/L} \stackrel{(37a)}{=} \left(\frac{1}{p_j}\right)^{Lp_j} = \prod_{j=1}^L p_j^{-p_j} . \quad (37b)$$

Thereby we get the entropy

$$\eta \stackrel{(36)}{=} k \ln \left( \prod_{j=1}^L p_j^{-p_j} \right) = - \sum_{j=1}^L k p_j \ln p_j . \quad (38a)$$

Now we can skip the inflexible assumption (37a), and may instead insert probabilities  $p_j$  which better fit the considered system, provided that

$$\sum_{j=1}^L p_j = 1 \quad (38b)$$

is respected. Jaynes [14] proved that Gibbs statistical formula (38) reproduces indeed the results of Clausius' entropy definition (3) under quite general conditions, provided that the probability distribution  $p_j$  is chosen appropriately.

#### 4. (id)-entropy

Consider a variable  $X$ , which can assume the  $n$  different discrete values  $\{x_1, \dots, x_n\}$ .  $X$  may for example be the result of throwing

a die (then  $n = 6$  for a conventional die), or the result of tossing a coin (then  $n = 2$ ), or the signal (or a sequence of signals) received from a telecommunication line.

It will be easier to understand Shannon's approach to information entropy, if we upfront define the *information content* of a particular value  $x_j$  of the variable  $X$ . The information content of a result with a fair die is  $3\times$  as high as the information content of a result with a fair coin, because the result of tossing the coin can be guessed in advance (before the coin is actually thrown and the result observed) with probability  $1/2$ , while the result of throwing the die can be guessed in advance only with probability  $1/6$ . Obviously the information content  $I(x_j)$  should be a function of  $p_j^{-1}$ , with  $p_j$  being the probability of  $X = x_j$ :

$$I(x_j) = \text{function}(p_j^{-1}) \quad (39a)$$

If two observations of the variable  $X$  are independent (for example the result  $x_a$  of throwing a die, and the result  $x_b$  of a later throw with the same die), then the information content of the combined results is additive:

$$I(x_a \cdot x_b) = I(x_a) + I(x_b) \quad (39b)$$

Thus  $I(x_j)$  must be a logarithmic function of  $x_j$ . Following Shannon, we choose the logarithm to the basis 2, and define — considering (39a) and (39b) —

$$I(x_j) = \log_2 p_j^{-1} \quad (40)$$

as information content of  $x_j$ .

In his “Mathematical Theory of Communication” [5], Shannon looked for a measure  $H(X)$  of the “choice” or “uncertainty” of an observation of  $X$  (“how difficult is it to guess the next  $x_j$ ”), for example the uncertainty of the next signal to be received from a

telecommunication line. The measure  $H(X)$  should meet these three requirements:

- \* It should be continuous in the  $p_j$ .
- \* If all  $p_j$  are equal (i. e.  $p_j = 1/n$ ), then  $H$  should be a monotonic increasing function of  $n$ , because “with equally likely events there is more choice, or uncertainty, when there are more possible events.”
- \*  $H(X)$  should have the same additive property (39b) as the information content  $I(x_j)$ .

Shannon proved that — besides an irrelevant constant factor — the only function which meets all three requirements is the mean value of the information contents  $I(x_j)$  of all values  $x_j$  which the variable  $X$  may assume:

$$H(X) = \sum_{j=1}^n p_j I(x_j) \stackrel{(40)}{=} - \sum_{j=1}^n p_j \log_2 p_j \quad (41)$$

Here  $H$  is the capital Greek letter  $H = \text{Eta}$ . Shannon called this quantity information-entropy, we call it in this article (id)-entropy. Shannon evaluated the (id)-entropy of discrete and continuous informations. In the latter case the sum in (41) becomes an integral. This is, however, merely a technical detail, and does not affect the conceptual basis of (id)-entropy.

The formal equivalence of (41) with Gibbs’ formulation (38a) of (td)-entropy is striking, as the constant factor  $k \ln 2$ , by which (38a) differs from (41), is clearly irrelevant in a fundamental consideration. But note that  $p_j$  in Gibb’s (td)-entropy refers to the probability of *microstates* which are compatible with the observed *macrostate*, while  $p_j$  in Shannon’s (id)-entropy refers to the probability of the *macrostates* which the observed system may assume.

To make this clear, and to justify the (d) = digital in the name (id)-entropy, and to illustrate further peculiar features of this type

of entropy, let's consider as an instructive example the (id)-entropy of a deck of 52 playing cards.<sup>6</sup> All 52 cards have different topsides, but identical backsides. Note that this is already a clearly digital interpretation. The backsides of the 52 cards certainly are not strictly identical, as could easily be proved by microscopic analysis. Like we may arbitrarily define in a digital interpretation that the differences of the seven graphs (1a) are irrelevant, and all seven graphs represent the letter A, we may in a digital interpretation arbitrarily define that the differences between the backsides of the cards are irrelevant. The 52 backsides of the cards are in our digital interpretation *identical by definition*.

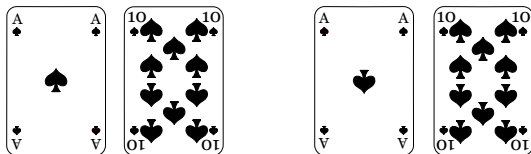


Fig. 2: Two cards with two different orientations

The cards have many properties: Seize, thickness, weight, material, ... The one and only property of this deck, which is of interest for our evaluation, is the order (i. e. the sequence), in which the 52 different cards are arranged. We discern face-up orientation from face-down orientation, and we assume that the long and short sides of the rectangular cards are aligned in the deck. But there is a further degree of freedom, see fig. 2, which we arbitrarily ignore *by definition*: The cards may be rotated by 180°. This is easily visible for cards with odd numbers (like the ace), and with some care (microscopic analysis, if needed) it can certainly be seen for all 52 cards.

The digital interpretation of states due to the arbitrary selective choice of “relevant” criteria is a characteristic of (id)- and (td)-

<sup>6</sup> This example owes much to Toffoli's 2016 review of information-entropy [15].

entropy, which has been pointed out in statement (31) above.

With the deck of cards, this sequence of actions is performed:

- ① We check the order of the cards, and note it on a piece of paper. Then the deck is put face-down onto the table, such that the card with position 1 is at bottom, and the card with position 52 is on top of the stack.
- ② One of the cards is picked out at about  $2/3$  of the stack height. We don't look very carefully from which exact position the card was taken, must have been somewhere in the range of position 30 through 40. Without looking at the face of the card, we put it back into the stack somewhere at about  $1/3$  of the stack height. Again we don't carefully check the new position of the card, certainly not below position 11, and certainly not above the (old) position 22. Due to this only imprecisely controlled position change of one card, we lost some information about the order of the deck.

We know for sure: At positions 1 through 11 and 41 through 52 the cards have not changed. Hence our list is certainly still correct with regard to these positions. And at positions 23 through 30 now are (in unchanged sequence) the cards which before have been at positions 22 through 29. But we have no idea which of the cards 30 through 40 has been picked (11 different cards are possible), and which exactly is it's new position (might be any of the (new) positions 12 through 22, i. e. 11 different positions possible).

Thus, if we want to keep track of the actual state (i. e. the actual positions of all cards) in the deck, the best we can do is to continue the bookkeeping with 121 different lists in parallel. One of these lists will be correct, and all 120 other lists will be wrong. But as long as we don't turn the cards face-up, we can not know which one is the true list. We must keep all 121 lists, if we want to be sure to have one correct list.

- ③ One card is picked, and replaced into the stack at a new position. But this time we carefully trace the action: The card is picked from position 27, and then is inserted above position 3. We update all 121 lists accordingly. Note that this time we don't need to increase the number of lists, because we didn't lose any information in this completely controlled action.
- ④ The deck is thoroughly shuffled, and then put back face-down onto the table. Now  $52!$  different lists are needed to document the actual state (i. e. the positions of the cards) of the deck. We know that only one list is correct, and all  $52! - 1$  other lists are wrong. But we do not know which one is the correct list.
- ⑤ Again one card is picked, and re-inserted at a new position. This time bookkeeping is simple: We don't need to do anything! With the  $52!$  lists, at which we arrived in step ④, we have already reached the maximum. One of these lists will be true, whatever further rearrangements of the order of cards may happen.
- ⑥ The card from position 37 is picked, and turned face-up. It's the jack of hearts. Then it is turned face-down again, and re-inserted at its previous position. This time the number of lists decreases: We only keep the  $51!$  lists which have the jack of hearts on position 37, and skip all other lists.
- ⑦ The deck is turned face-up. We skip all lists which don't match the actual order of cards, and only keep the one list which tallies with the actual state. Then we return the deck again face-down onto the table, without changing the order of cards.

We will use the letter  $L$  for the number of lists needed for complete bookkeeping of an incompletely known system, because this number replaces in (id)-entropy the number  $L$  of microstates, which are compatible with the observed macrostate, as defined in (td)-entropy.

Note the conceptual difference:

- \* (td)-entropy:  $L$  is the number of *microstates*, which are compatible with the completely known *macrostate* of the observed system.
- \* (id)-entropy:  $L$  is the number of *macrostates*, which are compatible with our incomplete knowledge about the *macrostate* of the observed system.

In the deck-of-cards example, we had

$$\begin{aligned} \textcircled{1} \rightarrow L = 1 & \quad \textcircled{2} \rightarrow L = 121 & \quad \textcircled{3} \rightarrow L = 121 & \quad \textcircled{4} \rightarrow \\ \textcircled{4} \rightarrow L = 52! & \quad \textcircled{5} \rightarrow L = 52! & \quad \textcircled{6} \rightarrow L = 51! & \quad \textcircled{7} \rightarrow L = 1 . \end{aligned} \quad (42)$$

$L$  is a natural number in the interval from 1 (complete knowledge) to  $L_{\max}$  = total number of states accessible to the system ( $L_{\max} = 52!$  in the deck-of-cards example).

The  $L$  of (id)-entropy increases whenever we loose information about the actual state of the system, and it decreases whenever we gain information about the actual state of the system. In contrast, in case of (td)-entropy the number  $L$  has a tendency to increase permanently, unless work is done from outside to decrease  $L$  (this is essentially the second law of thermodynamics). In thermodynamics, the state with  $L_{\max}$  is called “thermal equilibrium”.

To achieve the additive property (39b), Shannon [5] defined not  $L$  but

$$H = \log_2 L \quad (43)$$

as information entropy. In (37) and (38) it was demonstrated, that (43) is equivalent to

$$H \stackrel{(41)}{=} - \sum_{j=1}^L p_j \log_2 p_j \quad (44a)$$

$$\sum_{j=1}^L p_j = 1, \quad (44b)$$

if all of the  $L$  lists are true with same probabilities

$$p_j = \frac{1}{L} \quad \text{for } j = 1, 2, 3, \dots, L.$$

In the example of the deck of cards, however, the 121 lists, which we needed after step ②, certainly are *not* all true with same probability. For example the list which assumes that the card has been picked from position 35 and re-inserted above position 16 (that is in the centers of the possible ranges) is certainly more probably true than the list which assumes that the card has been picked from position 40 and re-inserted above position 11 (that is at the boundaries of the possible ranges). Thus in the example of the deck of cards, and in most other cases, Shannon's general formula (44) fits better than the particular formula (43).

Note the fundamental conceptual difference between (id)-entropy and physical quantities like mass or electric charge: After step ⑥ and before step ⑦, we ascribed to the deck of cards this (id)-entropy:

$$H \stackrel{(44)}{=} -51! \frac{1}{51!} \log_2 \frac{1}{51!} = \log_2 51! \quad (45a)$$

Then, in step ⑦, we turned the deck, kept only the one list which matched the actual order of cards, and put the deck again face-down onto the table. After step ⑦ we assigned to the deck the (id)-entropy

$$H \stackrel{(44)}{=} -\log_2 1 = 0. \quad (45b)$$



Before and after step ⑦ we assigned different (id)-entropies to the deck, even though the relevant property of the deck (i. e. the order of cards) did not change. In contrast, neither (ta)-entropy nor (td)-entropy change, unless there is an objective change of the considered system.

Not the order of the deck has changed, but our knowledge of the order of the deck has changed in step ⑦. If we have complete knowledge, then we need only 1 list, and the (id)-entropy is zero. The value of  $H > 0$  is a measure for the amount of information, which we would need (but actually don't have) for a unique description of the actual state of the considered system, i. e. a description by only 1 list.

(id)-entropy is not a quantity of the considered system. Instead (id)-entropy quantifies our missing information about the actual state of the system.

Consequently different observers may correctly assign completely different (id)-entropies to the identical system at the same point of time, if their knowledge about the system's actual state is different. (46)

A further important difference between (id)-entropy and both types of (t)-entropy is, that (t)-entropy tends to increase permanently (this is the second law of thermodynamics), while no such tendency exists in case of (id)-entropy. Well, there exists a faint similarity to the second law: If the state of a system is observed at some point of time, but in the sequel traced only theoretically, without further observations, then the (id)-entropy which we assign to that system increases, if the system develops according to stochastic rules, or it is constant, if the system develops according to deterministic rules. In no case will the (id)-entropy assigned to a merely theoretically traced system decrease. This faint similarity,

however, can certainly not be considered an equivalent counterpart to the powerful second law of thermodynamics. Instead the general rule is:

The (id)-entropy assigned to a system decreases whenever we gain information about that system, and it increases whenever we loose information about that system. Different from (t)-entropy, decrease of (id)-entropy is not restricted by a law of nature. (47)

A very useful application of the concept of (id)-entropy has been pointed out by Jaynes [16, p. 621]:

“Previously, one constructed a theory based on the equations of motion, supplemented by additional hypotheses of ergodicity, metric transitivity, or equal *a priori* probabilities, and the identification of entropy was made only at the end, by comparison of the resulting equations with the laws of phenomenological thermodynamics. Now, however, we can take entropy as our starting concept, and the fact that a probability distribution maximizes the entropy subject to certain constraints becomes the essential fact which justifies use of that distribution for inference.”

To illustrate the concept of turning (id)-entropy from a result of evaluations to a starting point of evaluations, Jaynes presents this example: We get the definitions of two different functions  $f(X)$  and  $g(X)$  of a variable  $X$  which can assume the discrete values  $\{x_1 \dots x_n\}$ , and are asked: What is the expectation value  $\langle g(X) \rangle$ ? To find out the precise answer

$$\langle g(X) \rangle = \sum_{j=1}^n p_j g(x_j) \quad (48a)$$

we would need to know all  $n$  values of the probability distribution  $p_j$ . But we get none of them. Instead we only get the value of  $\langle f(X) \rangle$ . Thus the only two informations onto which we can base a guess of  $\langle g(X) \rangle$  are

$$\langle f(X) \rangle = \sum_{j=1}^n p_j f(x_j) \quad (48b)$$

$$\sum_{j=1}^n p_j = 1 . \quad (48c)$$

(48b) and (48c) are two linearly independent informations, so we are missing merely  $n - 2$  informations for the precise answer (48a). Two informations are more than nothing. Hence we should not simply shrug shoulders, but be able to deliver an “educated best guess” of  $\langle g(X) \rangle$ . This guess will, of course, be based on our best guess of the probability distribution  $p_j$ . Jaynes [16, pp. 622, 623] comments:

“This problem of specification of probabilities in cases where little or no information is available, is as old as the theory of probability. Laplace’s ‘Principle of Insufficient Reason’ was an attempt to supply a criterion of choice, in which one said that two events are to be assigned equal probabilities if there is no reason to think otherwise. However, [...] this assumption may appear just as arbitrary as any other that might be made.

[...] our problem is that of finding a probability assignment which avoids bias, while agreeing with whatever information is given. [...] there is a unique, unambiguous criterion for the ‘amount of uncertainty’ represented by a discrete probability distribution, which agrees with our intuitive notions that a broad distribution represents more uncertainty than does a sharply peaked one, and

satisfies all other conditions which make it reasonable.<sup>7</sup> [...] Shannon proved] that the quantity which is positive, which increases with increasing uncertainty, and is additive for independent sources of uncertainty, is

$$H(p_1 \dots p_n) = -K \sum_i p_i \ln p_i, \quad (2-3)$$

where  $K$  is a positive constant.

[...] It is now evident how to solve the problem; in making inferences on the basis of partial information we must use that probability distribution which has maximum entropy subject to whatever is known. This is the only unbiased assignment we can make; to use any other would amount to arbitrary assumption of information which by hypothesis we do not have.”

This reasoning became known as the maximum-entropy-principle, or MaxEnt for brevity. To answer the question for  $\langle g(X) \rangle$  by MaxEnt, we must maximize the (id)-entropy

$$H(X) \stackrel{(41)}{=} - \sum_{j=1}^n p_j \log_2 p_j \quad (49)$$

due to variation of the  $p_j$  under the constraints (48b) and (48c). As usual, the constraints are included due to Lagrange multipliers  $\alpha$  and  $\beta$  into the variation of  $H$ :

$$-\delta H = 0 = \sum_{j=1}^n \delta(p_j \log_2 p_j) \stackrel{(48c)}{+} \sum_{j=1}^n \alpha \delta p_j \stackrel{(48b)}{+} \sum_{j=1}^n \beta \delta p_j f(x_j) \quad (50)$$

With  $\alpha = (\lambda - 1) \log_2 e$  and  $\beta = \mu \log_2 e$  this becomes

<sup>7</sup> Here Jaynes inserted a footnote, referencing to Shannon's article [5].

$$\begin{aligned}
 0 &= \sum_{j=1}^n \delta p_j \left( \ln p_j + p_j \frac{1}{p_j} + \lambda - 1 + \mu f(x_j) \right) \log_2 e \\
 \implies p_j &= e^{-\lambda - \mu f(x_j)} .
 \end{aligned} \tag{51}$$

This gives us indeed unique values for all  $p_j$ , because the two unknown multipliers  $\lambda$  and  $\mu$  can be determined from the two linearly independent equations

$$\langle f(X) \rangle \stackrel{(48b)}{=} \sum_{j=1}^n e^{-\lambda - \mu f(x_j)} f(x_j) \tag{52a}$$

$$\sum_{j=1}^n e^{-\lambda - \mu f(x_j)} \stackrel{(48c)}{=} 1 . \tag{52b}$$

I will not ponder the question, whether MaxEnt is a theorem, supported by a stringent proof, or merely a heuristic strategy. In any case it seems very plausible that this method leads indeed to a “probability assignment which avoids bias, while agreeing with whatever information is given.”

## 5. Conclusions

The three concepts of entropy, which are by today in use in physics, have been compared. All three are — within their appropriate application ranges — useful concepts of theoretical physics:

- \* The concept of (ta)-entropy can be consistently applied only within phenomenological thermodynamics. Extension to statistical evaluations results into problems like “Gibbs’ paradox”.
- \* The concept of (td)-entropy is useful for statistical evaluations of tangible objects.
- \* The concept of (id)-entropy is useful for statistical evaluations of informations, which may be — but do not need to be —

informations about the actual states of tangible objects.

The three entropy types differ in their relations to the “objective world out there”, which is assumed to exist independent of observations:

- \* (ta)-entropy changes, if and only if there is an objective change of the considered system.
- \* (td)-entropy changes only if there is an objective change of the considered system. The reverse is not true. Whether or not an objective change of the considered system implies a change of (td)-entropy does depend on arbitrarily chosen digital criteria, according to which the physicist assigned a state function to the system.
- \* Changes of (id)-entropy are not correlated with objective changes of the considered system. Instead (id)-entropy changes, whenever we gain or loose information about the considered system.

The three entropy types differ with regard to objective measurability:

- \* If different physicists analyze the same process, they measure in any case the same change of (ta)-entropy.
- \* If different physicists analyze the same process, they measure the same change of (td)-entropy only if they apply the same digital criteria for the definition of the state of the considered system.
- \* (id)-entropy is by construction (in general) different for different observers, hence not objectively measurable.

The three entropy types differ with regard to the notion of “states”:

- \* Changes of (ta)-entropy depend on the quotient  $Q/T$ , but not on the state assigned to the system.
- \* (td)-entropy is the larger, the larger the number of different *microstates* which are compatible with the *macrostate* assigned

to the system.

- \* (id)-entropy is the larger, the larger the number of different *macrostates* which are compatible with our knowledge of the actual *macrostate* assigned to the system.

The three entropy types differ with regard to the second law of thermodynamics:

- \* A decrease of (ta)- or (td)-entropy without compensating entropy increase of the environment is restricted by a — in case of (ta)-entropy deterministic, in case of (td)-entropy probabilistic — law of nature.
- \* No such restriction exists for a decrease of (id)-entropy.

(ta)-entropy may be called an “objective” quantity of the considered system, because it’s of no relevance for this quantity whether or not a physicist is looking. (id)-entropy, on the other hand, may be called a “subjective” quantity, which characterizes not objective facts but the knowledge of a physicist about objective facts.

The concept of (td)-entropy refuses to be squeezed into a simple objective / subjective scheme. In contrast to (id)-entropy, any change of (td)-entropy is strictly correlated with an objective change of the considered system, i. e. (td)-entropy has not a purely subjective character. And in contrast to (ta)-entropy, (td)-entropy is not uniquely determined by objective facts “out there”, i. e. (td)-entropy has not a purely objective character.

Gibbs’ (td)-entropy shares this peculiar inseparable amalgamation of subjective and objective elements with the state vector of quantum theory. In his Gifford lecture [17], Heisenberg characterizes the state vector (which he calls “probability function” in this lecture) as follows:

“The probability function combines objective and subjective elements. It contains statements about possibilities or better tendencies (‘potentia’ in Aristotelian philos-

ophy), and these statements are completely objective, they do not depend on any observer; and it contains statements about our knowledge of the system, which of course are subjective in so far as they may be different for different observers.”

The notable parallel with regard to the mixture of subjective / objective elements in Gibbs’ method of thermodynamics and quantum theory was well-known to the Copenhagen quantum physicists. Heisenberg reports (for example in [18, chap. 9]), that Bohr concluded many discussions on the intricate interplay of subjective and objective traits in quantum theory with remarks like “you can read all this in the writings of Gibbs.” With regard to that mixture, Jaynes [19] once complained that quantum theory is

“a peculiar mixture describing in part realities of Nature, in part incomplete human information about Nature — all scrambled up by Heisenberg and Bohr into an omelette that nobody has seen how to unscramble. Yet we think that the unscrambling is a prerequisite for any further advance in basic physical theory. For, if we cannot separate the subjective and objective aspects of the formalism, we cannot know what we are talking about”.

The demand for a nice separation of subjective and objective elements, however, is missing the true character of any state function. By construction, the state function must inevitably display an incomplete, “subjectively” chosen selection of “objective” parameters, which characterize the considered system. It is plainly impossible to compile a purely objective state function, because it is impossible to compile a complete state function. Quantum theory added the complication, that canonical conjugate variables are not precisely accessible at the same time.




But already in classical physics we can not avoid to restrict the state function — by arbitrarily, i. e. “subjectively” chosen digital















criteria — to a finite number of “relevant” objective parameters, and ignore all other (known, or yet unknown) objective parameters. We only need to replace “thermodynamics” by “quantum mechanics”, to make Gibbs’ remark “It is to states of systems thus incompletely defined that the problems of thermodynamics relate.” [3, part I, p. 228] a sentence, which could have been written by Bohr or Heisenberg.




It is a question of minor importance only, whether the decision, which parameters are considered relevant and which are considered irrelevant, is made arbitrarily or enforced by the impossibility to access canonical conjugate parameters at the same time. The essential fact is, that this decision can not be avoided. The Copenhagen quantum physicists had understood that by end of the nineteen-twenties. Gibbs had understood that half a century earlier, and it was exactly this insight which led him to the conception of (td)-entropy.

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