Interaction of 2-level systems and electromagnetic radiation

The interaction of quantum mechanical 2-level systems and electromagnetic radiation is outlined in detail. Rabi-oscillations and the dynamics of the Bloch vector are described for coherent and incoherent systems. Eventually we deal with Ramsey interferences and their practical use for precise time measurements by means of atomic clocks.

1. Rabi-oscillations of coherent 2-level systems

The theory of the interaction of quantum-mechanical 2-level systems and electromagnetic radiation, which will be presented in very detail in this article, has been worked out mainly by Isidor Rabi (1898 – 1988) and Felix Bloch (1905 – 1983). It is a semiclassical theory. In this context, semiclassical means that the 2-level system is treated quantum-mechanically, while the electromagnetic field, with which it is interacting, is treated classically. In state $|1\rangle$ the quantum system's energy is ε_1, in state $|2\rangle$ it's energy is $\varepsilon_2 > \varepsilon_1$. The system is irradiated with narrow-band electromagnetic radiation of frequency

$$\omega = (\varepsilon_2 - \varepsilon_1)/\hbar + \delta = \omega_2 - \omega_1 + \delta \ . \tag{1}$$

The detuning δ of the radiation versus the resonance frequency $\omega_2 - \omega_1$ may be larger, smaller, or equal to zero. We assume that the energies of all other excited states of the system are so high, that they can be safely ignored in the sequel.

Without radiation, the Hamilton operator of the 2-level system is

$$H_0 = |1\rangle \varepsilon_1 \langle 1| + |2\rangle \varepsilon_2 \langle 2| \ . \tag{2}$$

We confine ourselves to the electric dipole approximation, in which the magnetic field of the electromagnetic radiation is neglected, and the interaction energy is approximated by

$$H_W \overset{\text{dipole approximation}}{\approx} -\boldsymbol{d}\hat{\boldsymbol{E}} \cos(-\omega t) \ . \tag{3}$$

$\boldsymbol{d}$ is the electric dipole moment of the 2-level system, $\hat{\boldsymbol{E}}$ is the amplitude of the electric field, ω is the field's frequency, and t is representing time. As the wavelength of the radiation is much larger than the size of the 2-level system, the space coordinates of the electric field are ignored, and only it's time coordinate is considered.

The Hamilton operator of the interacting system can be expanded with regard to the vectors $|1\rangle$ and $|2\rangle$, which span the 2-level systems 2-dimensional Hilbert space:

$$\begin{aligned} H \overset{(2),(3)}{=} & |1\rangle \varepsilon_1 \langle 1| + |2\rangle \varepsilon_2 \langle 2| - |1\rangle \underbrace{\langle 1|\boldsymbol{d}\hat{\boldsymbol{E}}|1\rangle}_{0} \cos(-\omega t) \langle 1| - \\ & - |1\rangle \underbrace{\langle 1|\boldsymbol{d}\hat{\boldsymbol{E}}|2\rangle}_{\hbar\Omega_R^* \neq 0} \cos(-\omega t) \langle 2| - |2\rangle \underbrace{\langle 2|\boldsymbol{d}\hat{\boldsymbol{E}}|1\rangle}_{\hbar\Omega_R \neq 0} \cos(-\omega t) \langle 1| - \\ & - |2\rangle \underbrace{\langle 2|\boldsymbol{d}\hat{\boldsymbol{E}}|2\rangle}_{0} \cos(-\omega t) \langle 2| \ . \end{aligned} \tag{4}$$

Here the Rabi frequency

$$\Omega_R \equiv \frac{1}{\hbar} \langle 2|\boldsymbol{d}\hat{\boldsymbol{E}}|1\rangle = \frac{1}{\hbar} \Big(\langle 1|\boldsymbol{d}\hat{\boldsymbol{E}}|2\rangle \Big)^* \tag{5}$$

has been defined. In general, it is complex. Alternatively, some authors define the matrix elements of the system's electric dipole moment $\boldsymbol{d}$ onto the polarization axis of the electromagnetic radiation:

$$d_{12} \equiv \langle 1|\boldsymbol{d}\hat{\boldsymbol{e}}|2\rangle = \frac{\hbar\Omega_R^*}{|\hat{\boldsymbol{E}}|} \tag{6a}$$

$$d_{21} \equiv \langle 2|\boldsymbol{d}\hat{\boldsymbol{e}}|1\rangle = \frac{\hbar\Omega_R}{|\hat{\boldsymbol{E}}|} = d_{12}^* \tag{6b}$$

$$\hat{\boldsymbol{e}} \equiv \hat{\boldsymbol{E}}/|\hat{\boldsymbol{E}}|$$

The moduli of the matrix elements $\langle 2|\boldsymbol{d}\hat{\boldsymbol{E}}|1\rangle$ of absorption and $\langle 1|\boldsymbol{d}\hat{\boldsymbol{E}}|2\rangle$ of stimulated emission are equal. For the moment being, we stipulate that $\hat{\boldsymbol{E}}$ is so large that spontaneous emission and other relaxation mechanisms (e. g. excitations of vibration, or collisions with other systems) are negligible versus stimulated emission. Only in section 3 we will drop this assumption.

The 2-level system's generic state function

$$|\psi\rangle = c_1 e^{-i\omega_1 t}|1\rangle + c_2 e^{-i\omega_2 t}|2\rangle \tag{7}$$
$$c_1(t), c_2(t) \in \mathbb{C} \quad , \quad |c_1|^2 + |c_2|^2 = 1$$

evolves in time according to the Schrödinger equation:

$$i\hbar \frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle \tag{8}$$
$$i\hbar(\dot{c}_1 - i\omega_1 c_1)e^{-i\omega_1 t}|1\rangle + i\hbar(\dot{c}_2 - i\omega_2 c_2)e^{-i\omega_2 t}|2\rangle =$$
$$= \left(\varepsilon_1 c_1 e^{-i\omega_1 t} - \hbar\Omega_R^* \cos(-\omega t) c_2 e^{-i\omega_2 t}\right)|1\rangle +$$
$$+ \left(\varepsilon_2 c_2 e^{-i\omega_2 t} - \hbar\Omega_R \cos(-\omega t) c_1 e^{-i\omega_1 t}\right)|2\rangle$$

Here $\langle 1|1\rangle = \langle 2|2\rangle = 1$ and $\langle 1|2\rangle = 0$ has been used. As $|1\rangle$ and $|2\rangle$ are linearly independent, we get two differential equations for the probability amplitudes c_1 and c_2:

$$\begin{aligned} i(\dot{c}_1 - i\omega_1 c_1) &= \omega_1 c_1 - \Omega_R^* \cos(-\omega t) c_2 e^{-i(\omega_2-\omega_1)t} \\ \dot{c}_1 &= +i\,\frac{\Omega_R^* c_2}{2}\left(e^{-i(\omega_2-\omega_1+\omega)t} + e^{-i(\omega_2-\omega_1-\omega)t}\right) \\ i(\dot{c}_2 - i\omega_2 c_2) &= \omega_2 c_2 - \Omega_R \cos(-\omega t) c_1 e^{+i(\omega_2-\omega_1)t} \\ \dot{c}_2 &= +i\,\frac{\Omega_R c_1}{2}\left(e^{+i(\omega_2-\omega_1-\omega)t} + e^{+i(\omega_2-\omega_1+\omega)t}\right) \end{aligned}$$

Because of $\omega_2 - \omega_1 \approx \omega$, one exponential function in each equation is oscillating very fast as compared to the other. We skip the fast oscillating terms, this is called the “rotating wave approximation”. Furthermore we insert $\delta \overset{(1)}{=} \omega - (\omega_2 - \omega_1)$:

$$\dot{c}_1 = \frac{i\Omega_R^*}{2}\, e^{+i\delta t}\, c_2 \tag{10a}$$

$$\dot{c}_2 = \frac{i\Omega_R}{2}\, e^{-i\delta t}\, c_1 \tag{10b}$$

For the solution of this system of coupled first-grade differential equations, the literature recommends the ansatz

$$c_1 = \left(a_1 e^{+i\Omega t/2} + b_1 e^{-i\Omega t/2}\right) e^{+i\delta t/2} \tag{11a}$$

$$c_2 = \left(a_2 e^{+i\Omega t/2} + b_2 e^{-i\Omega t/2}\right) e^{-i\delta t/2} \tag{11b}$$

with indeterminate constants $a_1, b_1, a_2, b_2 \in \mathbb{C}$ and $\Omega \in \mathbb{R}$. The frequency Ω must be real, because otherwise one of the exponential functions in each equation would become infinite for $t \to \infty$, and thus the condition $|c_1|^2 + |c_2|^2 = 1$ could impossibly be met.

Inserting (11) into (10) gives

$$\begin{aligned}
\dot{c}_1 &= +\frac{i\delta}{2}\Big(a_1 e^{+i\Omega t/2} + b_1 e^{-i\Omega t/2}\Big)\, e^{+i\delta t/2} + \frac{i\Omega}{2}\, a_1 e^{+i\Omega t/2} e^{+i\delta t/2} - \\
&\quad - \frac{i\Omega}{2}\, b_1 e^{-i\Omega t/2} e^{+i\delta t/2} \\
&= i\,\Big(\frac{\Omega+\delta}{2}\, a_1 e^{+i\Omega t/2} - \frac{\Omega-\delta}{2}\, b_1 e^{-i\Omega t/2}\Big)\, e^{+i\delta t/2} \\
&= \frac{i\Omega_R^*}{2}\, e^{+i\delta t}\, \Big(a_2 e^{+i\Omega t/2} + b_2 e^{-i\Omega t/2}\Big)\, e^{-i\delta t/2} \qquad (12a) \\
\dot{c}_2 &= -\frac{i\delta}{2}\Big(a_2 e^{+i\Omega t/2} + b_2 e^{-i\Omega t/2}\Big)\, e^{-i\delta t/2} + \frac{i\Omega}{2}\, a_2 e^{+i\Omega t/2} e^{-i\delta t/2} - \\
&\quad - \frac{i\Omega}{2}\, b_2 e^{-i\Omega t/2} e^{-i\delta t/2} \\
&= i\,\Big(\frac{\Omega-\delta}{2}\, a_2 e^{+i\Omega t/2} - \frac{\Omega+\delta}{2}\, b_2 e^{-i\Omega t/2}\Big)\, e^{-i\delta t/2} \\
&= \frac{i\Omega_R}{2}\, e^{-i\delta t}\, \Big(a_1 e^{+i\Omega t/2} + b_1 e^{-i\Omega t/2}\Big)\, e^{+i\delta t/2}\ . \qquad (12b)
\end{aligned}$$

Thus (11) indeed solves the equations (10), if the conditions

$$\begin{aligned}
\frac{\Omega+\delta}{2}\, a_1 e^{+i\Omega t/2} - \frac{\Omega-\delta}{2}\, b_1 e^{-i\Omega t/2} &= \frac{\Omega_R^*}{2}\, \Big(a_2 e^{+i\Omega t/2} + b_2 e^{-i\Omega t/2}\Big) \\
\frac{\Omega-\delta}{2}\, a_2 e^{+i\Omega t/2} - \frac{\Omega+\delta}{2}\, b_2 e^{-i\Omega t/2} &= \frac{\Omega_R}{2}\, \Big(a_1 e^{+i\Omega t/2} + b_1 e^{-i\Omega t/2}\Big)
\end{aligned}$$

are met. These conditions must be met in particular at $t = 0$:

$$\frac{\Omega+\delta}{2}\, a_1 - \frac{\Omega-\delta}{2}\, b_1 = \frac{\Omega_R^*}{2}\, \Big(a_2 + b_2\Big) \qquad (14a)$$

$$\frac{\Omega-\delta}{2}\, a_2 - \frac{\Omega+\delta}{2}\, b_2 = \frac{\Omega_R}{2}\, \Big(a_1 + b_1\Big) \qquad (14b)$$

With

$$c_1(t=0) = a_1 + b_1 \qquad (15a)$$

$$c_2(t=0) = a_2 + b_2\ , \qquad (15b)$$

we get from (14)

$$a_1 = \frac{\Omega-\delta}{2\Omega}\, c_1(t=0) + \frac{\Omega_R^*}{2\Omega}\, c_2(t=0) \tag{16a}$$
$$b_1 = \frac{\Omega+\delta}{2\Omega}\, c_1(t=0) - \frac{\Omega_R^*}{2\Omega}\, c_2(t=0) \tag{16b}$$
$$a_2 = \frac{\Omega+\delta}{2\Omega}\, c_2(t=0) + \frac{\Omega_R}{2\Omega}\, c_1(t=0) \tag{16c}$$
$$b_2 = \frac{\Omega-\delta}{2\Omega}\, c_2(t=0) - \frac{\Omega_R}{2\Omega}\, c_1(t=0)\ . \tag{16d}$$

Now we stipulate the boundary condition, that the system is prepared in state $|1\rangle$ at time $t=0$:

$$|c_1(t=0)|^2 = 1 \quad\Longrightarrow\quad c_1(t=0) = e^{i\zeta}\ ,\ \zeta\in\mathbb{R} \tag{17a}$$
$$|c_2(t=0)|^2 = 0 \quad\Longrightarrow\quad c_2(t=0) = 0 \tag{17b}$$

With this boundary condition, (16) simplifies to

$$a_1 = +\frac{\Omega-\delta}{2\Omega}\, e^{i\zeta} \tag{18a}$$
$$b_1 = +\frac{\Omega+\delta}{2\Omega}\, e^{i\zeta} \tag{18b}$$
$$a_2 = +\frac{\Omega_R}{2\Omega}\, e^{i\zeta} \tag{18c}$$
$$b_2 = -\frac{\Omega_R}{2\Omega}\, e^{i\zeta}\ , \tag{18d}$$

and the solution (11) becomes

$$\begin{aligned} c_1 &= \left(+ \frac{\Omega-\delta}{2\Omega}\, e^{+i\Omega t/2} + \frac{\Omega+\delta}{2\Omega}\, e^{-i\Omega t/2}\right) e^{i\zeta+i\delta t/2} \\ &= \left(\cos(\Omega t/2) - \frac{i\delta}{\Omega}\,\sin(\Omega t/2)\right)\, e^{i\zeta+i\delta t/2} \end{aligned} \tag{19a}$$

$$c_2 = \Big(+ \frac{\Omega_R}{2\Omega}\, e^{+i\Omega t/2} - \frac{\Omega_R}{2\Omega}\, e^{-i\Omega t/2}\Big)\, e^{i\zeta - i\delta t/2}$$
$$= \frac{i\Omega_R}{\Omega}\, \sin(\Omega t/2)\, e^{i\zeta - i\delta t/2} \ . \qquad (19b)$$

The constant phase angle ζ is irrelevant, and remains indeterminate. To determine the unknown frequency Ω, we make use of the condition

$$|c_1|^2 + |c_2|^2 = 1 \ . \qquad (20)$$

Inserting (19) into (20) gives

$$\cos^2(\Omega t/2) + \frac{\delta^2}{\Omega^2}\, \sin^2(\Omega t/2) + \frac{|\Omega_R|^2}{\Omega^2}\, \sin^2(\Omega t/2) = 1 \ ,$$

and consequently

$$\frac{\delta^2}{\Omega^2} + \frac{|\Omega_R|^2}{\Omega^2} = 1 \quad \Longrightarrow \quad \Omega = \pm\sqrt{|\Omega_R|^2 + \delta^2} \ .$$

From (19) we see, that the sign of Ω may be chosen arbitrarily. We decide for

$$\Omega = +\sqrt{|\Omega_R|^2 + \delta^2} \ . \qquad (21)$$

Ω is called the generalized Rabi frequency.

Thus the probabilities of the two states are

$$|c_1(t)|^2 = \cos^2(\Omega t/2) + \frac{\delta^2}{\Omega^2}\, \sin^2(\Omega t/2) \qquad (22a)$$
$$|c_2(t)|^2 = \frac{|\Omega_R|^2}{\Omega^2}\, \sin^2(\Omega t/2) \ . \qquad (22b)$$

The function $\sin^2(\Omega t/2)$ is oscillating twice as fast as the function $\sin(\Omega t/2)$. The probabilities $|c_1(t)|^2$ and $|c_2(t)|^2$ are oscillating with frequency Ω between their min- and max-values. If the radiation is exactly tuned to the 2-level system's resonance frequency, i. e. if $\delta = \omega - (\omega_2 - \omega_1) = 0$, then the system is oscillating with the Rabi frequency $|\Omega_R|$. If the radiation is detuned, i. e. if $\delta \neq 0$, then the system is oscillating with the generalized Rabi frequency $\Omega = (21) > |\Omega_R|$.

At time

$$T_\pi \equiv \frac{\pi}{\Omega} = \frac{\pi}{\sqrt{|\Omega_R|^2 + \delta^2}} \tag{23}$$

$|c_2(t)|^2$ reaches first time it's maximum value

$$|c_2(T_\pi)|^2 = |c_2(t)|^2_{\max} = \frac{|\Omega_R|^2}{\Omega^2} = \frac{|\Omega_R|^2}{|\Omega_R|^2 + \delta^2} \ . \tag{24}$$

Only if $\delta = \omega - (\omega_2 - \omega_1) = 0$, $|c_2(t)|^2$ reaches the value 1 and $|c_1(t)|^2$ reaches the value 0. If $\delta \neq 0$, then the min and max values are $|c_2(t)|^2_{\max} < 1$ and $|c_1(t)|^2_{\min} > 0$.

Summary: If the system is at $t = 0$ in state $|1\rangle$, then it will be due to interaction with the radiation field at time $t > 0$ in state

$$\begin{aligned} |\psi\rangle &\overset{(7)}{=} c_1 e^{-i\omega_1 t}|1\rangle + c_2 e^{-i\omega_2 t}|2\rangle \\ c_1 &\overset{(19)}{=} \Big(\cos(\Omega t/2) - \frac{i\delta}{\Omega}\sin(\Omega t/2)\Big)\, e^{i(\zeta+\delta t/2)} \\ c_2 &\overset{(19)}{=} \frac{i\Omega_R}{\Omega}\sin(\Omega t/2)\, e^{i(\zeta-\delta t/2)} \\ \Omega &\overset{(21)}{=} +\sqrt{|\Omega_R|^2 + \delta^2} \ . \end{aligned} \tag{25}$$

At the end of a radiation pulse of duration $T_\pi = (23)$, simply called π-pulse for brevity, the system's state is

$$|\psi\rangle \overset{(25)}{=} -\frac{i\delta}{\Omega}\, e^{i(\zeta-\omega_1 t+\delta T_\pi/2)}\, |1\rangle + \frac{i\Omega_R}{\Omega}\, e^{i(\zeta-\omega_2 t-\delta T_\pi/2)}\, |2\rangle \ , \tag{26a}$$

resp. with exact tuning $\delta = \omega - (\omega_2 - \omega_1) = 0$ of the electromagnetic wave to the 2-level system in state

$$|\psi\rangle = e^{i(\zeta - \omega_2 t)}\,|2\rangle\ . \tag{26b}$$

Besides a phase change, the system will stay in this state as long as there is no further radiation. If a second π-pulse is applied immediately[1] after the first π-pulse, the system is forced back into the state

$$|\psi\rangle \overset{(25)}{=} -e^{i(\zeta - \omega_1 t + \delta T_\pi)}|1\rangle\ ,$$

which differs only by a phase factor from the initial state. If $|1\rangle$ is the system's state at $t = 0$, then a $\pi/2-$pulse, i. e. a radiation pulse of duration

$$T_{\pi/2} \equiv \frac{\pi/2}{\Omega} = \frac{\pi/2}{\sqrt{|\Omega_R|^2 + \delta^2}}\ , \tag{27}$$

will force it into the state

$$\begin{aligned}|\psi\rangle \overset{(25)}{=} \sqrt{\frac{1}{2}}\Big[\Big(1 - \frac{i\delta}{\Omega}\Big)\ e^{i(\zeta - \omega_1 t + \delta T_{\pi/2}/2)}|1\rangle +{}\\ + \frac{i\Omega_R}{\Omega}\ e^{i(\zeta - \omega_2 t - \delta T_{\pi/2}/2)}|2\rangle\Big]\ ,\end{aligned} \tag{28a}$$

resp. with exact tuning $\delta = \omega - (\omega_2 - \omega_1) = 0$ of the electromagnetic wave to the 2-level system into the state

$$|\psi\rangle \overset{(25)}{=} \sqrt{\frac{1}{2}}\Big(e^{i(\zeta - \omega_1 t)}|1\rangle + ie^{i(\zeta - \omega_2 t)}|2\rangle\Big)\ . \tag{28b}$$

If the system is evaluated in state (28b), then it will be found with probability $1/2$ in state $|1\rangle$, and with probability $1/2$ in state $|2\rangle$.

[1] In section 4 the effect of a time delay between two radiation pulses will be evaluated.

We emphasize again, that all results of this section and the next section are valid only under the precondition, that transitions $|2\rangle \rightarrow |1\rangle$ happen exclusively due to stimulated emission of radiation, i. e. that spontaneous emission and other relaxation mechanisms are negligible. Only in section 3 we will drop this precondition.

2. Density matrix and Bloch vector

The state (25) is a "pure" state, i. e. there is a well-defined phase relation between the two terms of (25).

Consider an operator A, which is representing an observable of the 2-level system. The expectation value of this observable is

$$\langle A\rangle_\psi = \langle\psi|A|\psi\rangle \quad \text{with } \psi = (25)\ . \tag{29}$$

Alternatively and equivalently, this expectation value can be computed by means of the density operator ρ:

$$\langle A\rangle_\psi = \operatorname{tr}(\rho A) \tag{30a}$$

$$\rho \equiv |\psi\rangle\langle\psi| \tag{30b}$$

$\operatorname{tr}(\rho A)$ is the trace of the operator product ρA. The advantage of the density operator is, that (30) — in contrast to (29) — is still applicable if ψ is not a pure state but a mixture, i. e. if the relative phase angle between the two terms of (25) is irregularly fluctuating. This case will be considered in section 3 .

In the density-matrix-formalism, the Schrödinger equation is replaced by the von Neumann-Liouville equation

$$\begin{aligned}
\dot\rho &\overset{(30b)}{=} |\dot\psi\rangle\langle\psi| + |\psi\rangle\langle\dot\psi| \\
&\overset{(8)}{=} -\frac{i}{\hbar}\Big(H|\psi\rangle\langle\psi| - |\psi\rangle\langle\psi|H \Big) \\
&\overset{(30b)}{=} -\frac{i}{\hbar}\,[H,\rho]\ .
\end{aligned} \tag{31}$$

The 2-level system's density operator is

$$\rho \overset{(30b)}{=} |\psi\rangle\langle\psi| \overset{(25)}{=} c_1c_1^*|1\rangle\langle 1| + c_2c_2^*|2\rangle\langle 2| + \\ + c_1c_2^*e^{+i(\omega_2-\omega_1)t}|1\rangle\langle 2| + c_2c_1^*e^{-i(\omega_2-\omega_1)t}|2\rangle\langle 1| \ . \tag{32}$$

As commonly done in the literature, we use the same sign ρ for the density operator and for the density matrix, and leave it to the reader's attention to discern what is meant in any case. The 2-level system's density matrix is defined by

$$\rho \equiv \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \equiv \begin{pmatrix} \langle 1|\rho|1\rangle & \langle 1|\rho|2\rangle \\ \langle 2|\rho|1\rangle & \langle 2|\rho|2\rangle \end{pmatrix} = \\ \overset{(32)}{=} \begin{pmatrix} c_1c_1^* & c_1c_2^*e^{+i(\omega_2-\omega_1)t} \\ c_2c_1^*e^{-i(\omega_2-\omega_1)t} & c_2c_2^* \end{pmatrix} \ . \tag{33}$$

Inserting (25), we get

$$\rho_{11} = \cos^2(\Omega t/2) + \frac{\delta^2}{\Omega^2}\sin^2(\Omega t/2) \tag{34a}$$

$$\rho_{22} = \frac{|\Omega_R|^2}{\Omega^2}\sin^2(\Omega t/2) \tag{34b}$$

$$\rho_{21} = \rho_{12}^* \tag{34c}$$

$$\rho_{12} = -\frac{\Omega_R^*}{\Omega}\sin(\Omega t/2)\Big(\frac{\delta}{\Omega}\sin(\Omega t/2) + i\cos(\Omega t/2)\Big)\cdot \\ \cdot e^{i(\zeta-\zeta+\omega t)} \ . \tag{34d}$$

If (25) wasn't a pure state but a mixture, with the phase angle ζ_1 of the first term varying irregularly versus the phase angle ζ_2 of the second term, then there would be the factor

$$\text{mean value}\Big(e^{\pm i(\zeta_1-\zeta_2)}\Big) = 0$$

instead of the factor $e^{i(\zeta-\zeta)} = 1$ in the off-diagonal elements of the density matrix, and consequently these elements would vanish.

The Bloch vector $\boldsymbol{V} \equiv (V_x, V_y, V_z)$ is defined by

$$V_x \equiv \rho_{21} e^{i\omega t} + \rho_{12} e^{-i\omega t} \tag{35a}$$
$$\stackrel{(34)}{=} -\frac{(\Omega_R + \Omega_R^*)\delta}{\Omega^2} \sin^2(\Omega t/2) + i\,\frac{\Omega_R - \Omega_R^*}{\Omega} \sin(\Omega t/2)\cos(\Omega t/2)$$
$$\stackrel{\text{if } \Omega_R \in \mathbb{R}}{=} -\frac{2\Omega_R \delta}{\Omega^2} \sin^2(\Omega t/2)$$
$$V_y \equiv i\rho_{21} e^{i\omega t} - i\rho_{12} e^{-i\omega t} \tag{35b}$$
$$\stackrel{(34)}{=} -i\,\frac{(\Omega_R - \Omega_R^*)\delta}{\Omega^2} \sin^2(\Omega t/2) - \frac{\Omega_R + \Omega_R^*}{\Omega} \sin(\Omega t/2)\cos(\Omega t/2)$$
$$\stackrel{\text{if } \Omega_R \in \mathbb{R}}{=} -\frac{2\Omega_R}{\Omega} \sin(\Omega t/2)\cos(\Omega t/2)$$
$$V_z \equiv \rho_{22} - \rho_{11} \tag{35c}$$
$$\stackrel{(34)}{=} \frac{|\Omega_R|^2 - \delta^2}{\Omega^2} \sin^2(\Omega t/2) - \cos^2(\Omega t/2)\ .$$

From now on until the end of this article, we assume that indeed

$$\Omega_R \in \mathbb{R}\ . \tag{36}$$

Then the time derivatives of the Bloch vector's components get the simple form

$$\dot{V}_x = -\frac{2\Omega_R \delta}{\Omega} \sin(\Omega t/2)\cos(\Omega t/2) = \delta V_y \tag{37a}$$
$$\dot{V}_y = -\Omega_R \Big(\cos^2(\Omega t/2) - \sin^2(\Omega t/2) \Big) = \Omega_R V_z - \delta V_x \tag{37b}$$
$$\dot{V}_z = \frac{\Omega_R^2 - \delta^2}{\Omega} \sin(\Omega t/2)\cos(\Omega t/2) + \Omega \cos(\Omega t/2)\sin(\Omega t/2) =$$
$$= \frac{2\Omega_R^2}{\Omega} \cos(\Omega t/2)\sin(\Omega t/2) = -\Omega_R V_y\ . \tag{37c}$$

Defining the frequency vector

$$\boldsymbol{F} \equiv (F_x, F_y, F_z) \equiv (-\Omega_R, 0, -\delta) \ , \tag{38}$$

the relations (37) can be written in compact form as a vector product:

$$\dot{\boldsymbol{V}} = \boldsymbol{F} \times \boldsymbol{V} \tag{39}$$

This equation resembles the dynamic equation

$$\dot{\boldsymbol{M}} = \gamma \boldsymbol{B} \times \boldsymbol{M}$$

of the magnetic moment $\boldsymbol{M}$ with gyromagnetic factor γ in the constant magnetic field $\boldsymbol{B}$. Equations (39) resp. (37) were introduced by Felix Bloch in 1946 [1].

It is obvious from (35) and (39),

* that $\boldsymbol{V}(t=0) = (0, 0, -1)$. This is reflecting our standard boundary condition (17), according to which the 2-level system is prepared at time $t = 0$ in the state $|1\rangle$.
* that the modulus of the Bloch vector $\boldsymbol{V}$ is constant, and on the surface of a sphere with radius $|\boldsymbol{V}| = 1$ at any time. This sphere is called the Bloch sphere. Note that this conclusion is valid only under the precondition, that transitions $|2\rangle \rightarrow |1\rangle$ happen exclusively due to stimulated emission of radiation, but not due to spontaneous emission, nor due to radiationless relaxation mechanisms. In section 3 we will drop that precondition.
* that the Bloch vector $\boldsymbol{V}$ is oscillating with frequency $|\boldsymbol{F}|$ on a cone around the vector $\boldsymbol{F}$.

In accord with (23) and (27) we define

$$T_a \equiv \frac{a}{\Omega} = \frac{a}{\sqrt{|\Omega_R|^2 + \delta^2}} \quad , \quad a \in \mathbb{R} \ . \tag{40}$$

Figure 1 displays the Bloch vector's position on the Bloch sphere at different times. The coordinate system is sketched on the left side.

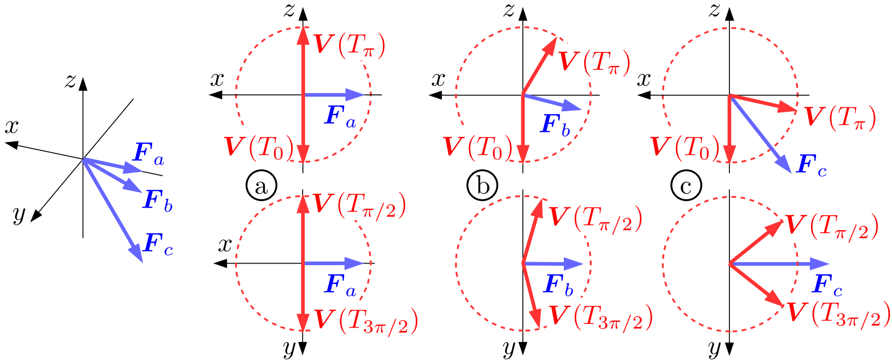

Fig. 1 : The Bloch sphere

The frequency vector $\boldsymbol{F} \overset{(38)}{=} (-\Omega_R, 0, -\delta)$ in any case is in the xz-plane. The x components of the three example vectors $\boldsymbol{F}_a$, $\boldsymbol{F}_b$, $\boldsymbol{F}_c$ are identical, namely $-\Omega_R$. In the case of $\boldsymbol{F}_a$, the detuning $\delta \overset{(1)}{=} \omega - (\omega_2 - \omega_1)$ of the exciting field versus the system's resonance frequency is zero, in the case of $\boldsymbol{F}_b$ the detuning is small, and in the case of $\boldsymbol{F}_c$ it is large. Due to our boundary condition (17), in any case $\boldsymbol{V}(T_0) = (0, 0, -1)$.

When the system is irradiated, $\boldsymbol{V}$ is rotating on a cone, whose axis is defined by $\boldsymbol{F}$. In case ⓐ the cone's angle is 180°, i. e. the cone is degenerated to a plane. In the cases ⓑ and ⓒ we have $\delta > 0$, and the cone's angle is smaller than 180°. In case of $\delta < 0$ (there is no example for this case in fig. 1), then the cone's angle is larger than 180°.

At time $t = T_{\pi/2}$, the Bloch vector $\boldsymbol{V}$ is in the plane spanned by the y-axis and the vector $\boldsymbol{F}$, see the three bottom sketches. Only if $\delta = 0$, this plane is identical to the xy-plane. At time $t = T_\pi$, the vector $\boldsymbol{V}$ is again in the xz-plane, and V_z reaches it's maximum value, which is larger than zero in case of $|\delta| < \Omega_R$, but smaller than zero in case of $|\delta| > \Omega_R$. Only if $V_z > 0$, the amplitude of $|2\rangle$ is larger than the amplitude of $|1\rangle$.

At time $t = T_{3\pi/2}$ the vector $\boldsymbol{V}$ is again in the plane spanned by the y-axis and $\boldsymbol{F}$. Eventually at time $t = T_{2\pi}$ the initial state is reached again, $\boldsymbol{V}(T_{2\pi}) = \boldsymbol{V}(T_0)$, the amplitude of $|2\rangle$ is zero, and the next cycle starts.

3. The incoherent 2-level system

Thus far we assumed that transitions $|2\rangle \rightarrow |1\rangle$ are caused exclusively by stimulated emission of radiation. But actually there exist alternatives: The energy of the excited state $|2\rangle$ can for example be dissipated due to spontaneous emission of radiation, or due to excitation of vibrations, or due to collisions with other systems. We introduce these additional relaxation channels phenomenologically into our model due to the insertion of relaxation times $0 < T_i < \infty$ with $i = 1$ or $i = 2$ into the time derivative of the density matrix:

$$\dot{\rho}_{11} = (\dot{34a}) + \frac{\rho_{22}}{T_1} \tag{41a}$$

$$\dot{\rho}_{22} = (\dot{34b}) - \frac{\rho_{22}}{T_1} \tag{41b}$$

$$\dot{\rho}_{21} = (\dot{34c}) - \frac{\rho_{21}}{T_2} \tag{41c}$$

$$\dot{\rho}_{12} = (\dot{34d}) - \frac{\rho_{12}}{T_2} \tag{41d}$$

The relaxation time T_1 is describing, how fast the state $|2\rangle$ decays due to spontaneous emission of radiation and other mechanisms, in addition to stimulated emission described by ($\dot{34b}$). The relaxation time T_2 is describing, how fast the system's coherence gets lost due to spontaneous emission and other mechanisms, and thus the pure state becomes a mixture. When the model is fitted to experiments, typically $T_2 \gtrapprox 2T_1$ is found.

With (41) the time derivative of the Bloch vector becomes

$$\dot{V}_x \overset{(37a)}{=} \delta V_y - \frac{\rho_{21}}{T_2}\, e^{i\omega t} - \frac{\rho_{12}}{T_2}\, e^{-i\omega t} \overset{(35)}{=} \delta V_y - \frac{V_x}{T_2} \tag{42a}$$

$$\dot{V}_y \overset{(37b)}{=} \Omega_R V_z - \delta V_x + i\,\frac{\rho_{21}}{T_2}\, e^{i\omega t} - i\,\frac{\rho_{12}}{T_2}\, e^{-i\omega t} =$$

$$\overset{(35)}{=} \Omega_R V_z - \delta V_x - \frac{V_y}{T_2} \tag{42b}$$

$$\dot{V}_z \overset{(37c)}{=} -\Omega_R V_y - \frac{2\rho_{22}}{T_1} \overset{(35)}{=} -\Omega_R V_y - \frac{V_z + 1}{T_1}\ . \tag{42c}$$

Due to the additional terms versus (37), $\dot{\boldsymbol{V}}$ can't any more be described by a vector product like (39). And we now are going to demonstrate, that the Bloch vector does not oscillate forever any more, but converges to a value with constant components and modulus $0 < |\boldsymbol{V}| < 1$. Using the definition

$$\widetilde{V}_i(t) \equiv \int_{\tau=0}^{t} \mathrm{d}\tau\, V_i(\tau) \quad \text{with } i = x, y, z\ , \tag{43}$$

we integrate (42) over time:

$$V_x(t) - \underbrace{V_x(t=0)}_{0} = \delta\widetilde{V}_y(t) - \frac{\widetilde{V}_x(t)}{T_2} \tag{44a}$$

$$V_y(t) - \underbrace{V_y(t=0)}_{0} = \Omega_R\widetilde{V}_z(t) - \delta\widetilde{V}_x(t) - \frac{\widetilde{V}_y(t)}{T_2} \tag{44b}$$

$$V_z(t) - \underbrace{V_z(t=0)}_{-1} = -\Omega_R\widetilde{V}_y(t) - \frac{\widetilde{V}_z(t) + t}{T_1} \tag{44c}$$

From the first two equations we get

$$\widetilde{V}_x = -V_x T_2 + \delta T_2 \widetilde{V}_y \tag{45a}$$

$$\widetilde{V}_x = -\frac{V_y}{\delta} + \frac{\Omega_R}{\delta}\,\widetilde{V}_z - \frac{\widetilde{V}_y}{\delta T_2} \tag{45b}$$

$$\widetilde{V}_y = \frac{\delta T_2^2 V_x}{\delta^2 T_2^2 + 1} - \frac{T_2 V_y}{\delta^2 T_2^2 + 1} + \frac{\Omega_R T_2}{\delta^2 T_2^2 + 1}\,\widetilde{V}_z \ . \tag{45c}$$

Insertion of (45c) into (44c) gives

$$\begin{aligned} V_z = &-1 - \frac{\delta \Omega_R T_2^2 V_x}{\delta^2 T_2^2 + 1} + \frac{\Omega_R T_2 V_y}{\delta^2 T_2^2 + 1} - \\ &- \frac{\Omega_R^2 T_2}{\delta^2 T_2^2 + 1}\,\widetilde{V}_z - \frac{\widetilde{V}_z}{T_1} - \frac{t}{T_1} \\ \widetilde{V}_z = &-\frac{\delta^2 T_1 T_2^2 + T_1}{\Omega_R^2 T_1 T_2 + \delta^2 T_2^2 + 1}\,V_z - \frac{\delta^2 T_1 T_2^2 + T_1}{\Omega_R^2 T_1 T_2 + \delta^2 T_2^2 + 1} - \\ &- \frac{\delta^2 T_1 T_2^2 + T_1}{\Omega_R^2 T_1 T_2 + \delta^2 T_2^2 + 1}\,\frac{\delta \Omega_R T_2^2 V_x}{\delta^2 T_2^2 + 1} + \\ &+ \frac{\delta^2 T_1 T_2^2 + T_1}{\Omega_R^2 T_1 T_2 + \delta^2 T_2^2 + 1}\,\frac{\Omega_R T_2 V_y}{\delta^2 T_2^2 + 1} - \frac{\delta^2 T_2^2 + 1}{\Omega_R^2 T_1 T_2 + \delta^2 T_2^2 + 1}\,t \ . \end{aligned}$$

For sufficiently large t, all other terms become negligible versus the last:

$$\widetilde{V}_z(t \to \infty) = -\frac{\delta^2 T_2^2 + 1}{\Omega_R^2 T_1 T_2 + \delta^2 T_2^2 + 1}\,t \tag{46a}$$

This is inserted into (45c):

$$\widetilde{V}_y(t \to \infty) = -\frac{\Omega_R T_2}{\Omega_R^2 T_1 T_2 + \delta^2 T_2^2 + 1}\,t \tag{46b}$$

Insertion into (45a) eventually gives

$$\widetilde{V}_x(t \to \infty) = -\frac{\delta \Omega_R T_2^2}{\Omega_R^2 T_1 T_2 + \delta^2 T_2^2 + 1}\, t \ . \tag{46c}$$

Thus the Bloch vector doesn't oscillate any more for large t, but converges to the stationary state

$$V_x(t \to \infty) = -\frac{\delta \Omega_R T_2^2}{\Omega_R^2 T_1 T_2 + \delta^2 T_2^2 + 1} \tag{47a}$$

$$V_y(t \to \infty) = -\frac{\Omega_R T_2}{\Omega_R^2 T_1 T_2 + \delta^2 T_2^2 + 1} \tag{47b}$$

$$V_z(t \to \infty) = -\frac{\delta^2 T_2^2 + 1}{\Omega_R^2 T_1 T_2 + \delta^2 T_2^2 + 1} \ . \tag{47c}$$

$-1 < V_z(t \to \infty) < 0$ according to the last equation. Thus in the steady state ($t \to \infty$), the probability for the system to be in state $|2\rangle$ is smaller than 0.5 in any case, even if $\delta = 0$, no matter how large the power of the radiation field may be. The Bloch vector's modulus in the steady state is

$$|\boldsymbol{V}|(t \to \infty) = \\ = \sqrt{\frac{T_2 \Omega_R^2 T_2 (\delta^2 T_2^2 + 1) + (\delta^2 T_2^2 + 1)^2}{\Omega_R^4 T_1^2 T_2^2 + 2 T_1 \Omega_R^2 T_2 (\delta^2 T_2^2 + 1) + (\delta^2 T_2^2 + 1)^2}} < 1 \ . \tag{48}$$

Using the common estimate $T_2 \gtrapprox 2T_1$, we see that $|\boldsymbol{V}|(t \to \infty) < 1$, and that the impact of the value of δ onto the value of $|\boldsymbol{V}|(t \to \infty)$ is marginal.

4. Ramsey fringes

"The second is the duration of 9 192 631 770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the ^{133}Cs atom." This is the definition of the second, as internationally agreed upon since 1967 . In Germany, it is the task of the Physikalisch-Technische Bundesanstalt in Braunschweig, to measure the second according to this definition, and provide it to the public.

For this purpose, the PTB lets a beam of ^{133}Cs atoms interact with a micro-wave field, whose frequency is determined by a voltage-controlled oscillator (VCO). At the start, the frequency of the VCO is adjusted to approximately 9.192 631 GHz. Thereby the atoms are oscillating between the state $|1\rangle$ (this is the sub-level $F_g = 3$ of the ground state) and the state $|2\rangle$ (this is the sub-level $F_g = 4$ of the ground state). The occupation numbers of the two states are

$$\rho_{11} \overset{(34a)}{=} \cos^2(\Omega t/2) + \frac{\delta^2}{|\Omega_R|^2 + \delta^2} \sin^2(\Omega t/2) \tag{49a}$$

$$\rho_{22} \overset{(34b)}{=} \frac{|\Omega_R|^2}{|\Omega_R|^2 + \delta^2} \sin^2(\Omega t/2) \ . \tag{49b}$$

Only if $\delta = 0$, ρ_{11} can reach the value 0, and ρ_{22} can reach the value 1. If $\delta \neq 0$, $\rho_{11} > 0$ and $\rho_{22} < 1$ at any time. Hence it would be sufficient "simply" to measure the amplitude of the Rabi-oscillations (49) and adjust the voltage-controlled oscillator such, that the amplitude becomes maximal. Then $\delta = 0$, we multiply the inverse of the oscillator frequency by 9 192 631 770, and have realized the second. In principle.

The disadvantage of this method is, that small deviations from the amplitude's maximum can not be observed very precisely, and hence the frequency with $\delta = 0$ can not be determined very accurately. In 1949, Norman Ramsey (1915 – 2011) suggested a better method [2]. The atomic clock "CS2" of the PTB has been built according to Ramsey's proposal as sketched in fig. 2 on the next page:

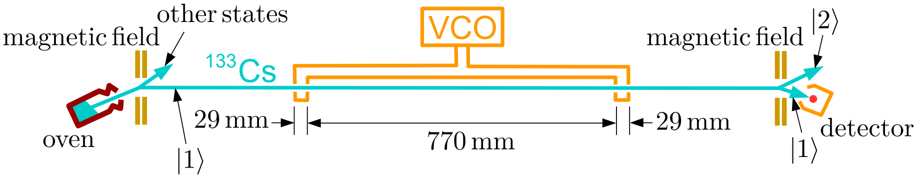

Fig. 2: The clock „CS2“ of the PTB

In a vacuum chamber, which is shielded against external magnetic fields, ^{133}Cs atoms are evaporated from an oven. An inhomogeneous magnetic field selects those atoms, which are in state $|1\rangle$ (this is the level $F_g = 3$ of the ground state).

There are about $1.3 \cdot 10^7$ atoms per second in this beam, which are moving with a velocity of about 95 m/s through the apparatus. They cross a waveguide of 29 mm width, in which they are excited by a microwave field of about 9.192 631 GHz. The microwave frequency is regulated and varied by means of a voltage-controlled oscillator (VCO). The intensity of the microwave field is adjusted such that the atoms, when crossing the waveguide, just are subject to a $\pi/2$-pulse.

Subsequently the atoms move during the time interval

$$T = \frac{770\,\text{mm}}{95\,\text{m/s}} \approx 8.1 \cdot 10^{-3}\,\text{s} \tag{50}$$

with no external field through the apparatus. Then they cross the waveguide again, hence are again subject to a $\pi/2$-pulse.

Eventually those atoms, which are excited into state $|2\rangle$, are separated from those atoms which are in state $|1\rangle$ by means of another inhomogeneous magnetic field. The atoms in state $|1\rangle$ are ionized by a hot wire, and then electrically detected.

Naively one might guess, that the two $\pi/2$-pulses in total have the same effect as one π-pulse. This guess is wrong, however, because the time $T = $ (50) between the two $\pi/2$-pulses, during which the atoms are moving free of external fields, basically changes the situation. For a detailed analysis, we start from the equations

$$\dot{V}_x \overset{(42)}{=} \delta V_y - \frac{V_x}{T_2} \tag{51a}$$

$$\dot{V}_y \overset{(42)}{=} \Omega_R V_z - \delta V_x - \frac{V_y}{T_2} \tag{51b}$$

$$\dot{V}_z \overset{(42)}{=} -\Omega_R V_y - \frac{V_z + 1}{T_1} \ . \tag{51c}$$

The setup is dimensioned such that

$$T = (50) \ll T_1 \qquad\qquad T \ll T_2 \tag{52a}$$

and hence a fortiori

$$T_{\pi/2} \ll T_1 \qquad\qquad T_{\pi/2} \ll T_2 \ . \tag{52b}$$

Therefore the last terms each in the three lines of (51) are negligible in good approximation. Furthermore the $\pi/2$-pulses

$$T_{\pi/2} \overset{(23)}{=} \frac{\pi/2}{\sqrt{|\Omega_R|^2 + \delta^2}}$$

are so strong, that

$$|\Omega_R|^2 \gg \delta^2 \ . \tag{52c}$$

We define a separate time scale for each single atom as follows: At time $t = -T_{\pi/2}$ the atom first time enters the waveguide, and leaves it again at time $t = T_0 = 0$. When the atom enters the waveguide, it's Bloch is

$$\boldsymbol{V}(-T_{\pi/2}) = (0, 0, -1) \ . \tag{53}$$

When the atom leaves the waveguide, it's Bloch vector is under the three presuppositions (51) in good approximation

$$\boldsymbol{V}(T_0) \approx (0, -1, 0) \; , \tag{54}$$

as sketched in fig. 3ⓐ and 3ⓑ on page 24. Subsequently (in the time interval from $t = T_0 = 0$ to $t = T$) the Bloch vector evolves according to (51):

$$\dot{V}_x = \delta V_y - \frac{V_x}{T_2} \overset{(52)}{\approx} \delta V_y \tag{55a}$$

$$\dot{V}_y = -\delta V_x - \frac{V_y}{T_2} \overset{(52)}{\approx} -\delta V_x \tag{55b}$$

$$\dot{V}_z = -\frac{V_z + 1}{T_1} \overset{(52)}{\approx} 0 \; . \tag{55c}$$

It is certainly plausible, that outside the radiation field all terms $\sim \Omega_R$ have been skipped from the Bloch vector. But it might be somewhat disturbing, that the terms $\sim \delta$ have been kept. Can we reasonably speak of the detuning δ, while there is no radiation which is detuned versus the atom's resonance frequency $\omega_2 - \omega_1$?

At this point we must recall the Bloch vector's definition (35):

$$V_x \equiv \rho_{21} e^{i\omega t} + \rho_{12} e^{-i\omega t} \tag{56a}$$

$$V_y \equiv i\rho_{21} e^{i\omega t} - i\rho_{12} e^{-i\omega t} \tag{56b}$$

$$V_z \equiv \rho_{22} - \rho_{11} \tag{56c}$$

$$\omega \equiv \omega_2 - \omega_1 + \delta$$

Due to the factors $e^{\pm i\omega t}$ there is an implicit dependence of V_x and V_y on δ, no matter whether the atom is subject to irradiation, or not. The Bloch vector does not describe the Cesium atom alone, but the total system atom & radiation. If we would describe the atoms and the radiation field by the methods of quantum field theory, then the state functions of the atoms and photons would get entangled due to the $\pi/2$-pulse. The entanglement persists when the atoms leave the waveguide. It ends only when the entangled system is subjected to a measurement. In Bloch's semiclassical formalism, the entanglement is represented by the implicit dependence of $\boldsymbol{V}$ on ω. Therefore it is reasonable, and consistent with our model, to keep the factor δ in (55) even if the atoms don't interact with the radiation field for some time. In contrast, the Rabi-frequency

$$\Omega_R \overset{(5)}{=} \frac{1}{\hbar}\langle 2|\boldsymbol{d}\hat{\boldsymbol{E}}|1\rangle = \frac{1}{\hbar}\Big(\langle 1|\boldsymbol{d}\hat{\boldsymbol{E}}|2\rangle\Big)^* \tag{57}$$

is clearly zero if the field $\boldsymbol{E}$ is zero at the position of the 2-level system, and thus has been correctly removed from (55).

Because of (52), $|\boldsymbol{V}|$ is in good approximation 1 at any time, and $\boldsymbol{V}$ is rotating during the time interval $0 \leq t \leq T$ in the xy-plane with frequency δ around the z-axis.

$$0 \leq t \leq T :$$

$$V_x(t) \approx -\sin(\delta t) \qquad V_x(t=0) = 0 \tag{58a}$$

$$V_y(t) \approx -\cos(\delta t) \qquad V_y(t=0) \approx -1 \tag{58b}$$

$$V_z(t) \approx 0 \tag{58c}$$

In fig. 3ⓒ and 3ⓔ, two arbitrary situations with different $\delta > 0$ are sketched. In case $\delta < 0$ (there is no example for this case in fig. 3) the Bloch vector is rotating in reverse direction around the z-axis, and in case $\delta = 0$ it is constantly $\boldsymbol{V} \approx (0, -1, 0)$.

The total rotation angle of $\boldsymbol{V}$ in the time interval T between the two pulses is

$$-\delta T = -\alpha - n \cdot 2\pi \quad , \quad n = 0, 1, 2, 3, \ldots \tag{59}$$

$$0 \leq \alpha < 2\pi \ .$$

In the time interval $T \le t \le T + T_{\pi/2}$, the Cesium atom is again subjected a $\pi/2$-pulse. This pulse rotates $\boldsymbol{V}$ from the xy-plane into the xz-plane, see fig. 3ⓓ and 3ⓕ. (Both are only approximations. If $\delta \neq 0$, then $\boldsymbol{V}$ is before the pulse not exactly, but almost, in the xy-plane. And it is after the pulse not exactly, but almost, in the xz-plane.) After this pulse, in good approximation

$$V_z \overset{(35)}{\equiv} \rho_{22} - \rho_{11} \approx \cos(\alpha) \ . \tag{60}$$

Thus in case $\alpha = 0$ the occupation probability of the state $|2\rangle$ is one, and the occupation probability of the state $|1\rangle$ is zero. In case $\alpha = \pi$ the occupation probability of state $|2\rangle$ is zero, and the occupation probability of state $|1\rangle$ is one.

In the time interval between $t = T + T_{\pi/2}$ and the detection, $V_z \overset{(35)}{=} \rho_{22} - \rho_{11}$ doesn't vary any more (in contrast to V_x and V_y). Hence the length of the time interval between $t = T + T_{\pi/2}$ and the detection is of no relevance with regard to the measurement result, provided that this time interval is very small versus T_1 and T_2.

In fig. 4, which is taken from [3], the measurement results are

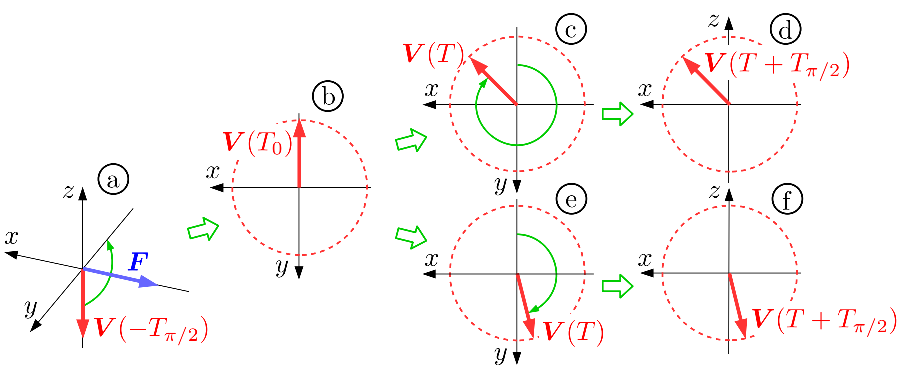

Fig. 3 : Der Bloch vector in Ramsey's method

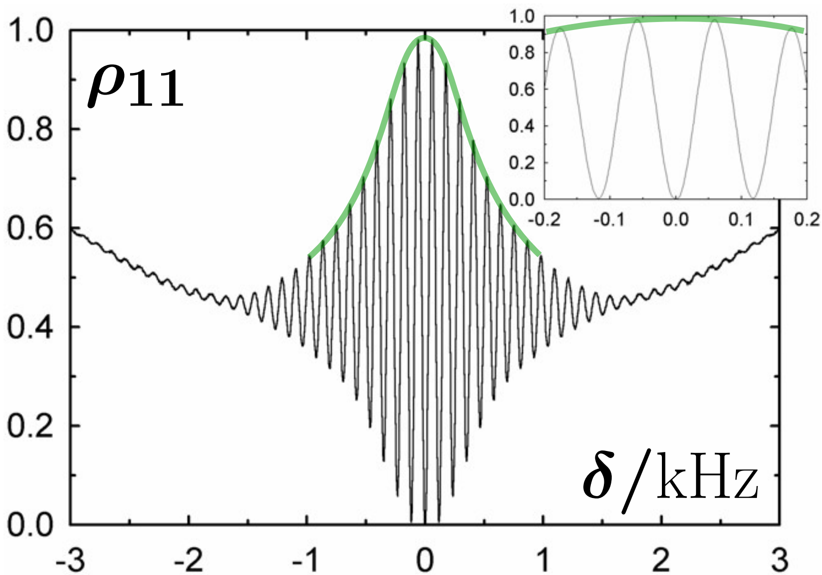

Fig. 4 : Ramsey fringes, observed with the clock "CS2" of the PTB (graph taken from [3])

displayed as function of δ. In the insert, the central fringes are displayed with enlarged frequency axis. As the clock "CS2" of the PTB measures ρ_{11} (but not ρ_{22}), the graph has at perfect resonance ($\delta = 0$) a minimum.

The advantage of Ramsey's method is clearly visible from this graph: The position of $\delta = 0$ can be read from the graph easily with an accuracy of about ± 10 Hz. If instead of the two $\pi/2$-pulses one single π-pulse would be applied (i. e. if T would be zero), then one would observe the curve indicated in green color, from which the position of $\delta = 0$ could be read only with an accuracy of about ± 100 Hz. Thus Ramsey's method increases the clock's accuracy by about a factor 10 .

Ramsey-interferences are quite special. In other interference experiments, light waves are interfering with light waves (resp. a photon is interfering with itself), or an electron wave is interfering with an electron wave (resp. an electron is interfering with itself), or a molecule wave is interfering with a molecule wave (resp. a molecule is interfering with itself). In contrast, fig. 4 shows the interference of a matter field (the Cesium atoms) and the electromagnetic radiation field. With the first $\pi/2$-pulse a well-defined phase relation between each Cesium atom and the radiation field is established. Subsequently the phase of the Cesium atoms evolves according to

$$\begin{aligned} |\psi\rangle &\overset{(28a)}{=} \sqrt{\frac{1}{2}}\Big[\Big(C_1 e^{-i\omega_1 t}|1\rangle + C_2 e^{-i\omega_2 t}|2\rangle\Big] \\ &= \sqrt{\frac{1}{2}}\Big[\Big(C_1 e^{+i(\omega_2-2\omega_1)t}|1\rangle + C_2 e^{+i\omega_1 t}|2\rangle\Big] \cdot e^{-i(\omega_2-\omega_1)t} \qquad (61) \\ C_1 &\equiv \Big(1-\frac{i\delta}{\Omega}\Big)\, e^{i(\zeta+\delta T_{\pi/2}/2)} \\ C_2 &\equiv \frac{i\Omega_R}{\Omega}\, e^{i(\zeta-\delta T_{\pi/2}/2)} \ . \end{aligned}$$

Note that C_1 and C_2 are constants. The exponential functions within the square brackets are oscillating very rapidly as compared to $e^{-i(\omega_2-\omega_1)t}$, and hence are merely "noise" with regard to the interference with the electromagnetic radiation oscillating at frequency $\omega \approx \omega_2 - \omega_1$.

At the same time, the phase of the radiation field evolves according to

$$\begin{aligned} \hat{\boldsymbol{E}} &\overset{(3)}{=} \hat{\boldsymbol{E}}\cos(-\omega t) \qquad (62) \\ \omega &\overset{(1)}{=} \omega_2 - \omega_1 + \delta \ . \end{aligned}$$

After the time T, when the atoms and the radiation meet again, the phase difference between (61) and (62) has added up to $-\delta T$, disregarding the “noise” within the square brackets of (61). Due to this phase difference we see in fig. 4 not the wide green curve, but the acute interference fringes, called Ramsey fringes.

References

[1] F. Bloch : *Nuclear Induction* , Phys. Rev. **70**, 460 – 474 (1946) , http://dx.doi.org/10.1103/PhysRev.70.460 or: http://baldwinlab.chem.ox.ac.uk/resources/bloch original paper.pdf

[2] N. F. Ramsey : *A Molecular Beam Resonance Method with Separated Oscillating Fields* , Phys. Rev. **78**, 695 (1950) http://dx.doi.org/10.1103/PhysRev.78.695 or: http://www2.mpq.mpg.de/~rnp/download/ls1415/papers/Ramsey_1950.pdf

[3] A. Bauch : *Caesium Atomic Clocks: Function, Performance and Applications* , (Physikalisch-Technische Bundesanstalt, Braunschweig, 2011) https://www.ptb.de/cms/fileadmin/internet/fachabteilungen/abteilung_4/4.4_zeit_und_frequenz/pdf/2003_Bauch_MST_CAC_VF_author_version.pdf