

Szilard's Theorem and Landauer's Principle

Are information entropy and thermodynamic entropy correlated?

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Szilard's theorem, published in 1929, and Landauer's principle, published in 1961, both postulate a relation between information entropy and thermodynamic entropy. In this article, both theorems are described in very detail, and both theorems are disproved.

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1. Thermodynamic entropy

The first law of thermodynamics says, that it is impossible to construct a “perpetuum mobile”, i. e. a continuously running machine whose energy output per machine cycle is higher than it's energy input. For centuries, ingenious inventors like Leonardo da Vinci (1452–1519) intensively tried to construct a perpetuum mobile, but all attempts failed.

This rule, extracted from experience, can be formally stated as

the first law of thermodynamics:

$$dU = \delta Q + dW \quad (1a)$$

Here U is the inner energy of the considered system, and Q and W are the heat and the work, respectively, which are fed² into the system.

It's characteristic for 19th century physics, to discern macroscopic interactions and microscopic interactions. Macroscopic interactions are mechanical work, and interactions with macroscopic electric, magnetic, and gravitational fields. Microscopic interactions are the exchange of heat.

Accordingly, the environment of the considered system is divided into a thermal bath, with which the system exchanges microscopic energy (i. e. the heat δQ in (1a)), and the macroscopic environment, with which the system exchanges macroscopic energy (i. e. the work dW in (1a)). Hence the first law can alternatively be stated for the closed total system, consisting of the system under consideration plus the thermal bath plus the macroscopic environment as

² We apply the convention, that work or heat, which is fed from outside into the system, gets a positive sign ($W > 0, Q > 0$), while work or heat, which the system feeds towards the environment, gets a negative sign ($W < 0, Q < 0$).

the first law of thermodynamics:

$$dU_{\text{closed}} = dU_{\text{system \& bath \& macrosc. env.}} = 0 \quad (1b)$$

The second law of thermodynamics, which — like the first law — has been concluded from experience, is even more restrictive. It says that it is impossible to construct a “perpetuum mobile of second kind”. This is³ a continuously running machine which extracts per cycle the energy Q in form of heat from *one* thermal bath, and converts it into mechanical energy² $W = -Q$. Instead there must exist a second thermal bath with temperature $T_2 < T_1$, and the machine which extracts the heat Q from the heat bath at T_1 can do only the mechanical work $W = -Q \cdot (T_1 - T_2)/T_1$, and must feed² the heat $q = -QT_2/T_1$ into the bath of lower temperature.

This can be formally stated as

the second law of thermodynamics:

$$\oint \frac{\delta Q}{T} \begin{cases} = 0 & \text{if the process is reversible} \\ < 0 & \text{if the process is irreversible} \end{cases} \quad (2)$$

In an irreversible cyclic process, the system dissipates more heat to the bath than it absorbs from the bath.

Clausius [2,3] defined the entropy S due it’s infinitesimal change

$$dS = \left. \frac{\delta Q}{T} \right|_{\text{rev}} = \begin{array}{l} \text{entropy change of a system, which} \\ \text{reversibly exchanges}^2 \text{ heat } Q \text{ with a} \\ \text{bath of temperature } T \end{array} \quad (3)$$

As there is no exchange of heat with the macroscopic environment, it’s entropy never changes:

³ In the words of Kelvin [1]: “It is impossible, by means of inanimate material agency, to derive mechanical effect from any portion of matter by cooling it below the temperature of the coldest of the surrounding objects.”

$$d S_{\text{macroscopic environment}} = 0 \quad (4)$$

Using the notion of entropy, a more detailed formulation of the second law is possible, which applies not only to cyclic processes, but to arbitrary processes:

The second law of thermodynamics:

$$d S_{\text{closed}} \stackrel{(4)}{=} d S + d S_{\text{bath}} \begin{cases} = 0 & \text{for reversible changes} \\ > 0 & \text{for irreversible changes} \end{cases} \quad (5a)$$

In words:

The entropy of a system can not decrease, unless that decrease is compensated or over-compensated by an increase of the entropy of the environment. (5b)

Don't be confused by the $>$ and $<$ signs in (2) and (5a), which result from our sign-convention².

Clausius [2, § 16] emphasized the difference

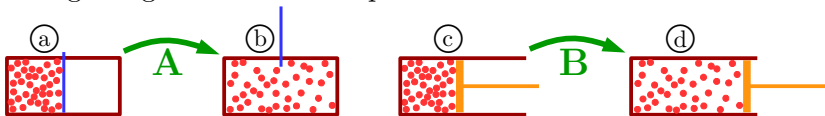
$$d \Omega = \left. \frac{\delta Q}{T} \right|_{\text{irreversible}} \neq \left. \frac{\delta Q}{T} \right|_{\text{reversible}} \stackrel{(3)}{=} d S \quad (6)$$

between the entropy S and the quantity Ω , which he named “not compensated transformation” (in 19th century German: „uncompensirte Verwandlung“). In an irreversible process, δQ may be completely different from $T d S$. In particular, δQ may be zero, even if $T d S \neq 0$.

A well-known example for $\delta Q = 0$ at $T d S \neq 0$ is the irreversible expansion of an ideal gas. Consider an ideal gas, consisting of N rare gas atoms enclosed in a vessel of volume V_1 , which is embedded in a thermal bath at temperature T . The equation of this ideal gas is

$$p_1 V_1 = N k T, \quad (7)$$

with p_1 being the pressure of the gas, and $k = 1.38 \cdot 10^{-23} \text{J/K}$ being the Boltzmann constant. Now the gas is expanded into a larger vessel of volume $V_2 = 2V_1$. This can be done irreversibly (process **A**) by pulling out a partition, or reversibly (process **B**) by letting the gas do work on a piston.



In process **B**, the expanding gas does the work²

$$-W = \int_{V_1}^{2V_1} dV p \stackrel{(7)}{=} NkT \int_{V_1}^{2V_1} \frac{dV}{V} = NkT \ln 2 . \quad (8)$$

The gas would cool down in process **B** unless the heat $Q = NkT \ln 2$ would be extracted from the bath. But there is no heat exchange between system and bath in process **A**, as experimentally proved by Gay-Lussac [4].

This difference of the two processes is easy to understand: The internal energy U of the gas is

$$U = N \frac{3kT}{2} , \quad (9)$$

with $kT/2$ being the mean energy per degree of freedom, which in case of rare gas atoms are the 3 degrees of freedom of translational motion. In process **A** the gas does no work, and the mean kinetic energy of an atom is not affected by pulling out the partition. Hence the first law of thermodynamics says in case of

process **A**:

$$dU = 0 \stackrel{(1a)}{=} \delta Q + \underbrace{dW}_0 \implies \delta Q = 0 \quad (10a)$$

When in process **B** the atoms are reflected from the outwards moving piston, they are slowed down, i. e. they do work² $W < 0$ onto the piston. If the system would be thermally isolated ($\delta Q = 0$), then the first law would give

$$\Delta U \stackrel{(1a)}{=} \underbrace{Q}_0 + \underbrace{W}_{<0} < 0 .$$

But due to the thermal bath, $T = \text{constant}$ is enforced, and consequently $U = \text{constant}$ due to (9). Therefore we get in case of

process **B**:

$$\Delta U \stackrel{(9)}{=} 0 \stackrel{(1a)}{=} Q + \underbrace{W}_{-NkT \ln 2} \implies Q = +NkT \ln 2 \quad (10b)$$

The gas absorbs from the bath the heat $Q \stackrel{(10b)}{=} NkT \ln 2 = T\Delta S$ in case of the reversible process **B**, but there is no heat exchange with the bath ($\delta Q \stackrel{(10a)}{=} 0$) in case of the irreversible process **A**, even though

$$\begin{aligned} S_{\text{a}} &= S_{\text{c}} \quad \text{and} \quad S_{\text{b}} = S_{\text{d}} \implies \\ \implies T\Delta S &= NkT \ln 2 \text{ is identical for processes } \mathbf{A} \text{ and } \mathbf{B} . \end{aligned}$$

This example confirms Clausius' assertion (6).

Boltzmann [5] and Gibbs [6] worked out the statistical interpretation of entropy. Boltzmann postulated

$$S = k \ln \Phi \quad (11)$$

for the entropy of an ideal gas in thermal equilibrium, which covers the phase space Φ . Gibbs generalized that formula to

$$S = -k \sum_j p_j \ln p_j \quad (12)$$

for the entropy of a system, which assumes with probabilities p_j various microstates j , all of which are compatible with the observed state of that system. The microstates are theoretically assumed states on a molecular scale, which is not accessible to direct experimental observation. Hence the observed state is sometimes for clarity called “macrostate”.

Actually there exists a basic, very important difference in the notions of entropy as defined by Clausius and Gibbs. In [7], that difference, and it’s far-reaching consequences, are explicated. For our present purpose — the discussion of Szilard’s theorem and Landauer’s principle —, however, that difference can be ignored. But the difference inbetween thermodynamic entropy (be it in the definition of Clausius or Gibbs) on the one hand, and information entropy on the other hand, will turn out to be essential in our below considerations.

2. Information entropy

Consider a variable X , which can assume the n different discrete values $\{x_1, \dots, x_n\}$. X may for example be the result of throwing a die (then $n = 6$ for a conventional die), or the result of tossing a coin (then $n = 2$), or the signal received from a telecommunication line.

We now want to define the *information content* of a particular value x_j of the variable X . The information content of a result with a fair die is $3\times$ as high as the information content of a result with a fair coin, because the result of tossing the coin can be guessed in advance (before the coin is actually thrown and the result observed) with probability $1/2$, while the result of throwing the die can be guessed in advance only with probability $1/6$. Obviously the information content $I(x_j)$ should be a function of p_j^{-1} , with p_j being the probability of $X = x_j$:

$$I(x_j) = \text{function}\left(\frac{1}{p_j}\right) \quad (13a)$$

If two observations of the variable X are independent (for example the result x_a of throwing a die, and the result x_b of a later throw with the same die), then the information content of the combined results is additive:

$$I(x_a \cdot x_b) = I(x_a) + I(x_b) \quad (13b)$$

Thus $I(x_j)$ must be a logarithmic function of x_j . In his “Mathematical Theory of Communication” [8], published in 1948, Claude Elwood Shannon (1916–2001) chose the logarithm to the basis 2. Considering (13a) and (13b), *information content* may be defined as follows:

$$I(x_j) = \log_2\left(\frac{1}{p_j}\right) = -\log_2 p_j \quad (14)$$

Furthermore Shannon [8] defined the *information entropy* of the variable X as the mean value of the information contents $I(x_j)$ of all values x_j which the variable X does assume with probability p_j :

$$H(X) = \sum_{j=1}^n p_j I(x_j) \stackrel{(14)}{=} -\sum_{j=1}^n p_j \log_2 p_j \quad (15)$$

Here H is the Greek capital letter Eta. Famously it was John v. Neumann, who in a discussion with Shannon suggested to name the quantity (15) entropy, “because anyway nobody knows what entropy is.” Indeed the similarity of the formulas for Gibb’s thermodynamic entropy (12) and Shannon’s information entropy (15) is striking.⁴

⁴ In this article, the word “information” is always explicitly stated in case of information entropy. When the word “entropy” is used without an explicit specifier “information” or “thermodynamic”, then always thermodynamic entropy is meant.

3. Szilard's theorem

In 1929, Leo Szilard (1898–1964) published an article [9], in which he tried to demonstrate a deep relation between the second law of thermodynamics and the accumulation of information. His assertion may appropriately be called

Szilard's theorem:

Any measurement and storage of the result causes per bit of measurement result minimum the entropy decrease (16a)

$$\Delta S = -k \ln 2 < 0$$

of measured object & measurement instrument & memory.

According to the second law of thermodynamics, an entropy reduction of minimum $\Delta S = -k \ln 2$ of measured object & measurement instrument & memory must be compensated by an entropy increase of minimum

$$\Delta S_{\text{bath}} = +k \ln 2 > 0 \tag{16b}$$

of the environment, i. e. at temperature T minimum the heat²

$$-Q = kT \ln 2 \tag{16c}$$

must be dissipated to the thermal bath. According to the first law, the energy (16c) must come from the work

$$W = kT \ln 2 , \tag{16d}$$

which minimum needs to be done per bit of measurement result, to accomplish the measurement.

Szilard did not explicitly formulate this theorem, but from the context of his article there can be no doubt that (16) is exactly what he wanted to point out.

To motivate and explicate this idea, Szilard considered three examples: A self-measuring gas, a perpetual mobile of second kind, and a measuring instrument with thermal memory. A general theorem like (16) can of course not be proved by the presentation of some few examples. Indeed we will see that only the self-measuring gas really supports (16), while the perpetual mobile on closer scrutiny turns to an argument against (16), and the thermal memory can easily be modified to an argument against (16). In the sequel, Szilard's three example machines are presented and discussed one by one.

3.1. The self-measuring gas

To demonstrate a relation between information and entropy⁴, Szilard considered a gas of atoms, which have — besides their three translational degrees of freedom — an internal degree of freedom, which can assume the two discrete values X^+ or X^- . This property is changing stochastically, such that an atom with X^+ changes at an unpredictable point of time to X^- , and at a later unpredictable point of time back to X^+ , and so on. While it is impossible to predict the point of time, at which a particular atom changes the X property, the time constant of the property change for a statistically huge number of atoms can of course be precisely determined. We will use the Greek character τ for the time constant of a full cycle from X^+ to X^- and back to X^+ .

The probability to observe an atom at some point of time with property X^+ is p^+ , and the probability to observe it with property X^- is p^- , with $p^+ + p^- = 1$.

Attached to each atom is a tiny measurement instrument and a 1-bit memory. Upon command of an observer, the instrument measures whether the atom is in state X^+ or X^- . If the result is X^+ , the memory is set to Y^+ . If the result is X^- , the memory is

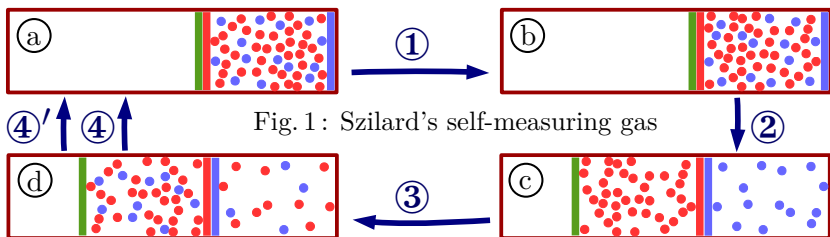


Fig. 1: Szilard's self-measuring gas

set to Y^- .

Szilard assumed to have a huge number N of these “molecules” available, each molecule consisting of one atom, one measuring instrument, and one memory. The molecules form an ideal gas, which is enclosed in a vessel, as sketched in figure 1. Red dots symbolize atoms which at this point of time are in state X^+ , blue dots symbolize atoms which are in state X^- . While “red” molecules are permanently changing to “blue”, and “blue” molecules are changing to “red”, the fraction N^+/N of “red” molecules deviates only negligibly from p^+ , and the fraction N^-/N of “blue” molecules deviates only negligibly from p^- at any time, because N is a huge number.

The gas is enclosed between movable partitions, whose positions are steered from outside. The blue partition right of the gas is permeable for molecules in state X^- , i. e. the “blue” molecules, but impermeable for molecules in state X^+ , i. e. the “red” molecules. The red partition left of the gas is permeable for molecules in state X^+ , but impermeable for molecules in state X^- . The movable green partition is impermeable for all molecules. The volume left of the green partition is evacuated. The vessel is embedded into a thermal bath of temperature T .

Now Szilard considered this cyclic process:

- (a) Long ago, very many time constants τ before the start of the cycle, the outside observer has ordered a measurement. Upon

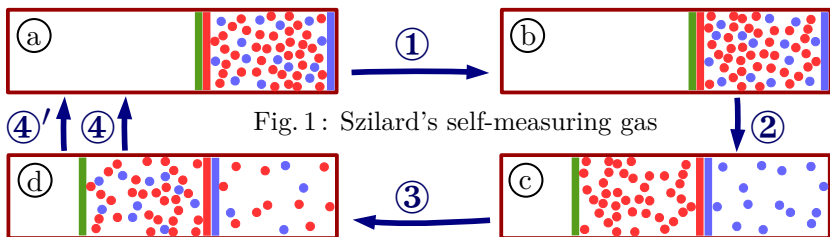


Fig. 1: Szilard's self-measuring gas

this order, each measurement instrument has measured the state of its atom (X^+ or X^-), and set its memory accordingly to Y^+ or Y^- . Thus immediately after that measurement, $N^+ = p^+N$ molecules have been in state X^+ and their memories were set to Y^+ , while $N^- = p^-N$ molecules have been in state X^- and their memories were set to Y^- . The atom state X and the memory state Y of each molecule had been perfectly correlated at that point of time.

Since then, almost all atoms have many times switched back and forth between X^+ and X^- , while the Y values of the memories have not changed since the measurement. Thus there are now still $N^+ = p^+N$ molecules in state X^+ and $N^- = p^-N$ molecules in state X^- , and N^+ memories in state Y^+ and N^- memories in state Y^- . But the correlation between the X and the Y of each molecule is completely lost. This is the state (a), from which the cyclic process starts.

- ① In the first process step, the outside observer orders a new measurement. This measurement changes the entropy of the gas. To see that, consider the probabilities, with which a single molecule might have the properties $\{X, Y\}$ before and after the measurement:

properties	probability before measurement	probability after measurement
$\{X^+, Y^+\}$	$p^+ p^+$	p^+
$\{X^+, Y^-\}$	$p^+ p^-$	0
$\{X^-, Y^+\}$	$p^- p^+$	0
$\{X^-, Y^-\}$	$p^- p^-$	p^-

Before the measurement, the molecules could be in four different $\{X, Y\}$ states, while immediately after the measurement they can be in only two different $\{X, Y\}$ states. According to Gibbs' entropy formula (12), there is in course of the measurement the entropy reduction

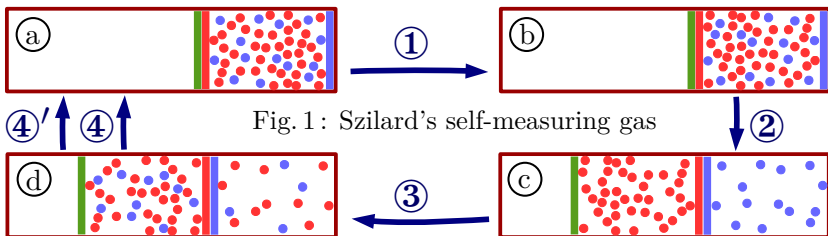
$$\begin{aligned}
 \Delta S &= Nk \left(-p^+ \ln p^+ - p^- \ln p^- + \right. \\
 &\quad \left. + p^+ p^+ \ln(p^+ p^+) + p^+ p^- \ln(p^+ p^-) + \right. \\
 &\quad \left. + p^- p^+ \ln(p^- p^+) + p^- p^- \ln(p^- p^-) \right) = \\
 &= Nk \left(-p^+ \ln p^+ - p^- \ln p^- \right) + \\
 &\quad + 2p^+ (p^+ \ln p^+ + p^- \ln p^-) + \\
 &\quad + 2p^- (p^+ \ln p^+ + p^- \ln p^-) \Big) = \\
 &= Nk \left(p^+ \ln p^+ + p^- \ln p^- \right) < 0 \tag{17a}
 \end{aligned}$$

of the gas. Here $p^+ + p^- = 1$ has been used. According to the second law, the entropy decrease (17a) of the gas must be compensated by an entropy increase

$$\Delta S_{\text{bath}} \geq -Nk(p^+ \ln p^+ + p^- \ln p^-) > 0 \tag{17b}$$

of the bath. To supply the energy needed for that entropy increase, minimum the work

$$W \geq NkT(p^+ \ln p^+ + p^- \ln p^-) > 0 \tag{17c}$$



must be done, to accomplish the measurement. Where does that energy come from? As we want the process fig. 1 to be cyclic, it wouldn't be useful to equip the tiny measurement devices with batteries, which would be exhausted after some cycles. Let's assume instead, that the outside observer supplies — together with the measurement order — the necessary energy (17c) due to microwave transmission.

- ② Immediately after the measurement the green partition (which is impermeable for all molecules) is moved left, and synchronously the blue partition (which is permeable for blue molecules, but impermeable for red molecules) is moved left at same speed. Thereby the sub-ensemble of red molecules is moved left with unchanged volume V_0 , while the sub-ensemble of blue molecules stays with unchanged volume V_0 at it's previous position. The movement is stopped when the blue partition touches the red partition. No work needs to be done to accomplish the shift, nor does the shift induce any change of temperature or entropy.⁵

⁵ To see this clearly, imagine that — without changing the volume V_0 — the red gas is by means of a semipermeable partition compressed into the left half of this volume, and the blue gas is by means of a semipermeable partition compressed into the right half of this volume, such that we get N^+ red molecules and no blue molecules compressed into the volume $V_0/2$ left of the semipermeable partitions, and N^- blue molecules and no red molecules compressed into the volume $V_0/2$ right of the semipermeable partitions. To

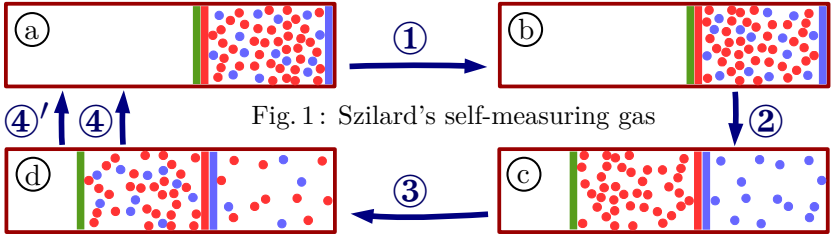


Fig. 1: Szilard's self-measuring gas

- Ⓒ If the shift is completed within a small time interval $\Delta t \ll \tau$ after the measurement, then immediately after the shift there are $N^+ = p^+ N$ molecules in the left volume, which (almost) all are in the “red” state X^+ and have memory content Y^+ . And there are $N^- = p^- N$ molecules in the right volume, which (almost) all are in the “blue” state X^- and have memory content Y^- .
- Ⓓ The third process step is merely a waiting time interval $\Delta t \gg \tau$. During this time, the atoms in both volumes change back and forth between X^+ and X^- , while their memory contents (i. e. the Y -values) don't change. There is a significant

achieve this reversible demixing, the work

$$\Delta W \stackrel{(22)}{=} kTN^+ \int_{V_0}^{V_0/2} \frac{dV}{V} + kTN^- \int_{V_0}^{V_0/2} \frac{dV}{V} \quad (18a)$$

must be done. After the demixing, each gas is reversibly expanded to volume V_0 , i. e. to state Ⓒ. Thereby the gas does the work

$$-\Delta W \stackrel{(22)}{=} kTN^+ \int_{V_0/2}^{V_0} \frac{dV}{V} + kTN^- \int_{V_0/2}^{V_0} \frac{dV}{V} = -(18a). \quad (18b)$$

Thus the overall net exchange of energy between gas and environment is zero, and there is indeed no change of entropy in process step Ⓓ.

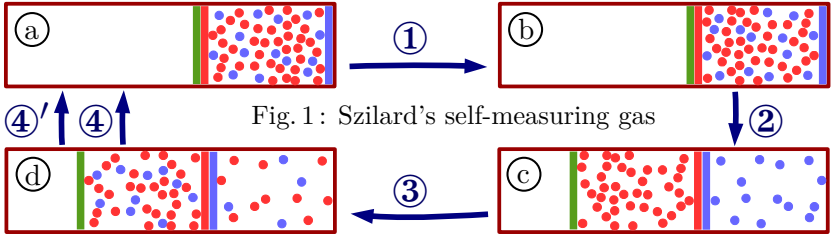


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loss of order:

properties	left volume probability in		right volume probability in	
	state (c)	state (d)	state (c)	state (d)
$\{X^+, Y^+\}$	1	p^+	0	0
$\{X^-, Y^+\}$	0	p^-	0	0
$\{X^+, Y^-\}$	0	0	0	p^+
$\{X^-, Y^-\}$	0	0	1	p^-

Consequently the entropy increases. Again this entropy change during process step (3) can be computed by means of Gibbs' entropy formula (12):

$$\begin{aligned}
 \Delta S &= \underbrace{-N^+ k \left(p^+ \ln p^+ + p^- \ln p^- + \overbrace{1 \ln 1}^0 + \overbrace{0 \ln 0}^0 \right)}_{\text{left volume}} - \\
 &\quad \underbrace{-N^- k \left(p^+ \ln p^+ + p^- \ln p^- + 0 \ln 0 + 1 \ln 1 \right)}_{\text{right volume}} = \\
 &= -Nk \left(p^+ \ln p^+ + p^- \ln p^- \right) > 0 \tag{19}
 \end{aligned}$$

Note that this increase of entropy exactly compensates the entropy decrease (17a) due to the measurement in step (1).

- (4) At start of the last process step, a further semipermeable partition, which is permeable for molecules with memory content Y^- , but impermeable for molecules with memory content Y^+ ,

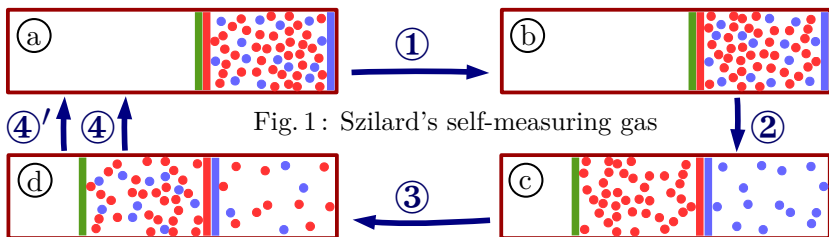


Fig. 1: Szilard's self-measuring gas

is vertically inserted directly right of the two semipermeable partitions. And directly left of the semipermeable partitions a further semipermeable partition is vertically inserted, which is permeable for molecules with memory content Y^+ , but impermeable for molecules with memory content Y^- . Then the two partitions, which are semipermeable for X^+ resp. X^- are vertically pulled out. The change of partitions doesn't need energy, and does not change any parameter of the system.

Remember that *all* molecules right of the semipermeable partitions have memory content Y^- , and *all* molecules left of the semipermeable partitions have memory content Y^+ . Now the partition, which blocks Y^+ but not Y^- , and the very left (green) partition, which blocks all molecules, are shifted right at same speed, until they touch the right vessel wall respectively the third partition. Thereby the Y^+ gas is shifted towards right at unchanged volume V_0 , and mixed into the Y^- gas, which as well keeps it's unchanged volume V_0 .

As both the Y^+ gas and the Y^- gas are ideal gases, and their volumes V_0 are constant during the mixing, no work is done during the mixing, there is no heat exchange with the bath, and no change of entropy. (That could be double-checked with a quite similar computation as presented in footnote 5 for process step ②.) At the end of process step ④, state (a) is reached again, and the process cycle is closed.

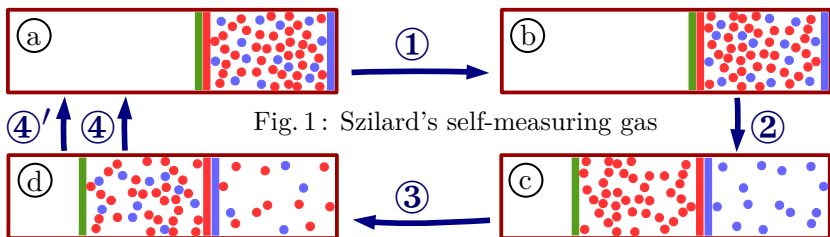


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There are no entropy changes in steps ② and ④ of this cyclic process, and the entropy changes of steps ① and ③ exactly cancel. If we had overlooked the entropy reduction due to the measurement in step ①, then $\oint dS$ would be different from zero, even though the process is clearly cyclic. Hence Szilard considered the self-measuring gas example a support for his theorem (16).

Let's for comparison compute the entropy balance, if the measurement in process step ① is skipped. That modification will not affect the entropy increase in process step ③, because only the probabilities for X^+ and X^- change during that process step and thereby contribute to the entropy change $\Delta S = (19)$.

But there is now a problem with process step ④. Because no measurement has been made and stored in step ①, this time the molecules with Y^+ are not gathered in the left volume, and the molecules with Y^- are not gathered in the right volume. Instead Y^+ and Y^- can be found in both volumes with probabilities p^+ and p^- , respectively. Therefore this time the gas can not be transferred to state ① by means of semipermeable partitions. The only method to restore the gas to state ① is this:

- ④' All semipermeable partitions are replaced by impermeable partitions. Remember that due to process step ① there are N^+ molecules in the left volume, and N^- molecules in the right volume. Now the left volume is compressed from V_0 to

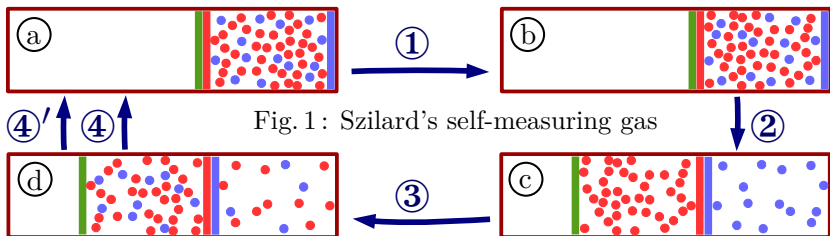


Fig. 1: Szilard's self-measuring gas

p^+V_0 . This requires the work²

$$W \stackrel{(22)}{=} -kTN^+ \int_{V_0}^{p^+V_0} \frac{dV}{V} = -kTN^+ \ln p^+ . \quad (20a)$$

The right volume is compressed from V_0 to p^-V_0 . This requires the work

$$W = -kTN^- \ln p^- . \quad (20b)$$

The total work needed for the compression is

$$W = (20a) + (20b) = -kTN(p^+ \ln p^+ + p^- \ln p^-) \quad (20c)$$

This work decreases the entropy of the gas by

$$\Delta S = kN(p^+ \ln p^+ + p^- \ln p^-) < 0 , \quad (20d)$$

and the gas transfers to the bath the heat²

$$Q = T\Delta S = kTN(p^+ \ln p^+ + p^- \ln p^-) < 0 . \quad (20e)$$

Thereby the entropy of the bath increases by

$$\Delta S_{\text{bath}} = -(20d) = -kTN(p^+ \ln p^+ + p^- \ln p^-) > 0 . \quad (20f)$$

The total gas volume now is V_0 , and the pressure is the same in both volumes. We only need to pull-out the partition between the volumes to close the cycle, with no further change of entropy.

Note that the entropy reduction of the gas, the work to be done, and the heat dissipated to the bath in step ④' all are equal to the respective values (17) due to the measurement in the original step ①.

Szilard explained that it seemed unlikely to him that the entropy decrease due to a measurement should depend on the particular values of p^+ and p^- . Considering that for arbitrary $p^+ = 1 - p^-$

$$-p^+ \ln p^+ - p^- \ln p^- \leq -\frac{1}{2} \ln \frac{1}{2} - \frac{1}{2} \ln \frac{1}{2}, \quad (21)$$

he decided to insert $\ln 2$ into (16a).

3.2. A perpetuum mobile of second kind

While the self-measuring gas example is supporting Szilard's theorem (16), severe problems turn up with his next example: The perpetuum mobile of second kind. He constructed that machine such, that an entropy reduction due to measurement is needed, to cure an alleged violation of the second law of thermodynamics. We will see, however, that actually that measurement is *not* correlated with any change of thermodynamic entropy.

The perpetuum mobile of second kind is sketched in fig. 2 on the next page in a slightly improved form as suggested by Bennet [10]. If this machine would be functional (of course it is not), it would work like this:

- Ⓐ The machine cycle starts from state Ⓐ. One single rare-gas atom is trapped in a vessel of volume V_m . The vessel is embedded into a thermal bath with temperature T . Movable

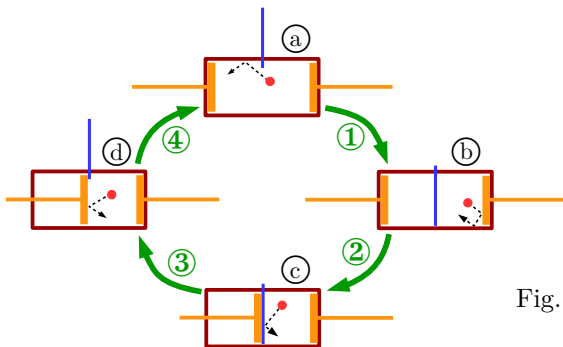


Fig. 2: Szilard's perpetual mobile of second kind

pistons can be shifted from left and right into the vessel. The atom exerts a pressure $p_{(a)}$ onto the pistons and onto the walls of the vessel, which can be computed from the equation of an ideal gas, consisting of 1 atom:

$$p_{(a)} \stackrel{(7)}{=} \frac{kT}{V_m} \quad (22)$$

- ① In the first process step, a partition is inserted, which divides the vessel into 2 cavities of size $V_m/2$ each. It's of no relevance whether the atom is trapped by chance in the left or in the right cavity. No (or only negligible) energy is required to shift-in the partition.⁶ As the volume of the ideal gas (consisting of the 1 rare gas atom) is changed without doing mechanical work on the gas or extracting work from the gas, the atom's kinetic energy, and hence the temperature of the gas, is not changed. But there is a change of the volume, which is accessible to the 1-atom-gas, and the pressure, which it exerts onto the vessel walls, the partition, and the pistons:

⁶ In this analysis, we assume van der Waals forces to be negligible. Later we will argue that this is a misleading over-simplification.

$$p_{(b)} = \begin{cases} kT/(V_m/2) = 2p_{(a)} & \text{in the cell with the atom} \\ 0 & \text{in the empty cell} \end{cases} \quad (23)$$

Thus the mean value over both cells is $\overline{p_{(b)}} = p_{(a)}$.

To find the entropy change, we must do the computation *as if* this process step would be done reversibly, even though it actually is done irreversibly. That is, we must assume that the gas would be confined to the left or right half vessel by slowly pushing in a piston. Thereby the work²

$$W = - \int_{V_m}^{V_m/2} dV p(V) \stackrel{(22)}{=} -kT \int_{V_m}^{V_m/2} \frac{dV}{V} = +kT \ln 2$$

would be done. Due to heat exchange with the bath, the mean kinetic energy of the atom

$$U_{\text{atom}} = \frac{3kT}{2}$$

is constant. Hence we get from the first law

$$\Delta U_{\text{atom}} = Q + W = 0 ,$$

and thereby this entropy change of the 1-atom gas:

$$\Delta S = \frac{Q}{T} = \frac{-W}{T} = -k \ln 2 < 0 . \quad (24)$$

(24) is as well the entropy change in the actually irreversible process step ①. But the Q and W showing up in the previous equations do not exist in the irreversible process step. Thus the entropy decrease (24) is not compensated by an entropy increase of the bath, clearly violating the second law (5). This strange fact makes Szilard's machine a perpetuum mobile of

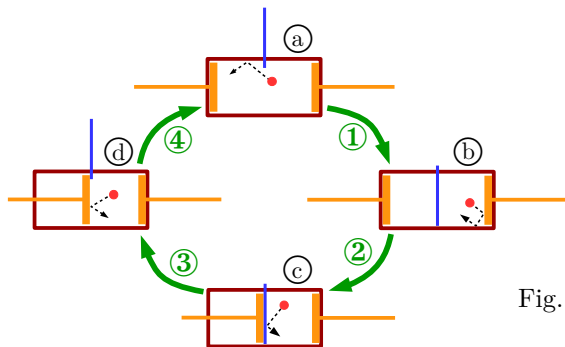


Fig. 2: Szilard's perpetual mobile of second kind

second kind.

- ② In the second process step, a small force

$$f \ll p_{(b)}A ,$$

with A being the area of the pistons, is exerted from outside onto *both* pistons, trying to shift them into the vessel. One of the pistons does not move, and consequently no work is done, because the small force f can not overcome the pressure $p_{(b)}$. The other piston is moved inside, until it touches the partition. No (or only negligible) work is done with this piston, because there is no (or only negligible) resistance against it's inside move. And there is no change of entropy in this process step.

- ③ In the third process step, appropriate mechanisms are attached to *both* pistons⁷ outside the vessel, which will lift loads against

⁷ To extract work from Szilard's original machine [9], the user needs to know upfront in which of the two compartments the atom is trapped. That's of course a faulty design. Bennet, though he suggested [10] the improved form of Szilard's machine, for unknown reasons still believed in error that an upfront measurement of the atom's position would be needed to extract energy from the machine, and — most important for Bennet's strange interpretation of this process — that the result of that measurement must be stored in a memory, and later be erased from that memory, at the expense of energy.

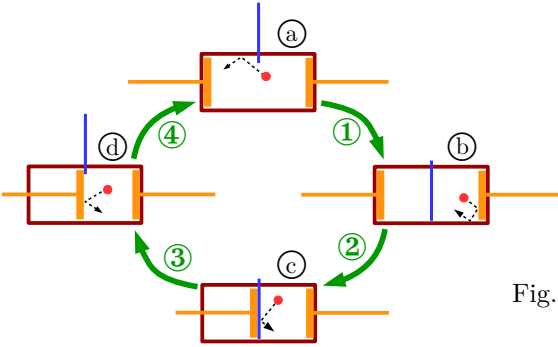


Fig. 2: Szilard's perpetual mobile of second kind

the earth's gravitational field, provided the pistons should be pushed towards the outside. Then the partition is pulled-out. This requires no (or only negligible) energy, and there is no change of entropy in this process step.

- ④ In the last process step, one of the pistons is moved due to the pressure exerted by the atom, and lifts the load of mass M up to height h . The useful mechanical work done by the machine is²

$$-W = Mgh = \int_{V_m/2}^{V_m} dV p(V) \stackrel{(22)}{=} kT \int_{V_m/2}^{V_m} \frac{dV}{V} = kT \ln 2 ,$$

with g being the gravitational acceleration. This energy comes from the heat

$$Q = -W = kT \ln 2 \quad (25)$$

which the atom absorbs from the thermal bath during process step ④.

As process step ④ is done reversibly, the entropy change of the 1-atom gas is

$$\Delta S = \frac{Q}{T} \stackrel{(25)}{=} +k \ln 2 > 0 , \quad (26a)$$

and the entropy change of the bath is

$$\Delta S_{\text{bath}} = -(26a) = -k \ln 2 < 0 . \quad (26b)$$

Note that $Q = W = 0$ in each of the process steps ①, ②, ③. Only in process step ④ the heat transfer $Q = -W > 0$ happens. Furthermore the entropy increase (26a) exactly compensates the entropy decrease (24), while the entropy decrease (26b) of the bath is not compensated.

Thus the machine does exactly, what is impossible according to the second law: It is a cyclic working machine, which in each cycle absorbs heat $Q > 0$ from a bath and converts it into usable work $-W = Q > 0$, *even though there exists no temperature difference between the bath, the machine, and the rest of the environment.* Stated in terms of entropy: This machine continuously decreases the entropy of the closed system gas & bath by $\Delta S_{\text{gas \& bath}} \stackrel{(26b)}{=} -k \ln 2$ per cycle.

Here Szilard saw an argument for his theorem (16a), which assigns an entropy reduction to *any* measurement. There is indeed a measurement in the cyclic process fig. 2: After the partition has been inserted in process step ①, we do not yet know whether the atom has been trapped in the left or in the right cavity.

We get that information due to process step ②, which clearly is a measurement and storage of the result. When a small force is applied to both pistons, but only one of them yields and moves in, then indeed the position of the atom has been measured, with the result that it must be in the other cavity. Of course we do not need⁷ this knowledge to successfully drive the perpetuum mobile and extract useful work from it. But the measurement has anyway been done, and the result (left piston in or right piston in) has

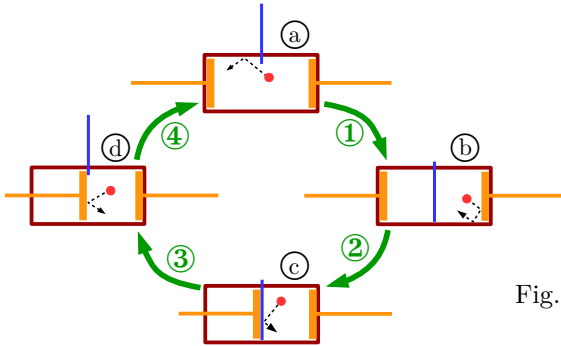


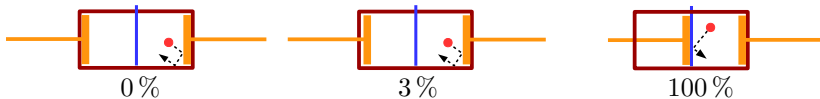
Fig. 2: Szilard's perpetual mobile of second kind

objectively come into existence, and is stored by the moved-in position of the piston. It's of no relevance whether anybody looks and notices the result.

Note the difference to the case of the self-measuring gas. There we computed — by means of standard textbook thermodynamics — an entropy reduction of the gas due to the measurement and storage of the result in process step ①. But pushing the piston into the empty cell in process step ② of the perpetual mobile fig. 2 is a measurement and storage of the result, which does clearly *not* imply an entropy change according to textbook thermodynamics.

According to Szilard's theorem (16a), however, there must be an entropy decrease $\Delta S = -k \ln 2$ due to this measurement and result storage. Consequently, if Szilard's theorem should be correct, then — due to the second law of thermodynamics — the work $W = kT \ln 2$ must be done in course of the measurement, and thereby the heat² $-Q = kT \ln 2$ is dissipated into the bath. This work just compensates the work done by the machine in process step ④, and the related heat extraction from the bath. Therefore the machine displayed in fig. 2 is no perpetual mobile of second kind, and the second law of thermodynamics is “saved”, if Szilard's theorem (16a) should be correct.

A severe problem of Szilard's theorem becomes visible, however, when process step ② is analyzed in detail. The three sketches show Szilard's machine after



of this process step have been completed. Let's assume that we can clearly discern which piston moves, and which not, as soon as one piston has moved 1% of it's full stroke. We could stop the process at this moment, and still would have 100% of the measurement result, i. e. we would know for sure whether the atom is trapped in the left or in the right cavity. Thus the measurement proper (including storage of the result!) is completed after 1% of the piston stroke, no additional information can be gathered during the remaining 99% of process step ②. Consequently, if we had a sufficiently sensitive calorimeter at our disposal, we would have registered 100% of the heat $kT \ln 2$, which is dissipated according to Szilard's theorem due to the measurement, within the first 1% of this process step. No further heat would be dissipated during the remaining 99% of the piston stroke.

The first law of thermodynamics must be preserved. This means that the dissipated heat does not come for free out of nothing. Instead — different from our previous assumption that the piston of the empty cavity can be moved with doing almost no work — the work $W_{\text{empty}} = kT \ln 2$ must be done to move this piston. And all of this work must be done during the measurement, i. e. within the first percent of the full piston stroke. With $L_0 = (V_m/2)/A$ being the full length of the piston stroke, and A being the area of the piston, the work

$$W_{\text{empty}} = \int_0^{1\% \cdot L_0} dL F_{\text{empty}} = 1\% \cdot L_0 \cdot \bar{F}_{\text{empty}} = kT \ln 2 \quad (27a)$$

$$\bar{F}_{\text{empty}} = \frac{100}{L_0} kT \ln 2 \quad (27b)$$

must be done, with F being the force resisting the inward move of the piston, and \bar{F} being the mean value of F during the first percent of the stroke. But as soon as the measurement result is available, F shrinks to zero, and the remaining 99% of the stroke can be done with no (or only negligible) work. Lets compare this work with the work that must be done to move the piston of the other cavity (the cavity with the atom) 1% in:²

$$W_{\text{atom}} = -kT \int_{V_m/2}^{99\% \cdot V_m/2} \frac{dV}{V} = -kT \ln 0.99 \quad (27c)$$

$$\bar{F}_{\text{atom}} = -\frac{100}{L_0} kT \ln 0.99 \stackrel{(27b)}{\approx} \frac{\bar{F}_{\text{empty}}}{69} \quad (27d)$$

\bar{F}_{empty} is almost 69 times as large as \bar{F}_{atom} ! Consequently, when the force applied to both pistons is slowly increased, then — contrary to our previous assumption — the piston of the cavity with the atom will yield first, but not the piston of the empty cavity. But . . . stop! Once we have learned which piston yields first, the 1% move of this piston gives us the full measurement result. According to Szilard's theorem, we can not get the measurement result cheaper than by the work $kT \ln 2$. Consequently neither \bar{F}_{atom} nor \bar{F}_{empty} can be smaller than $100 kT \ln 2 / L_0$.

This situation is strange, to say the least. And it becomes even stranger, if we consider this scenario: Imagine that due to a leak the atom unfortunately has suddenly disappeared, and the vessel is empty in process state [a](#). Now we perform process step [2](#). Both pistons will move in and touch the partition. From this result we get the information about the disaster. This information clearly is a measurement result. Consequently we will not get this

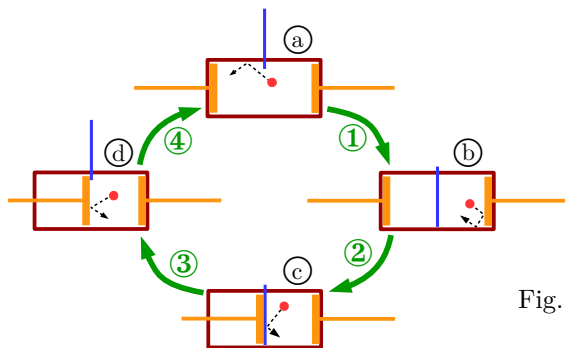


Fig. 2: Szilard’s perpetual mobile of second kind

information for free, but must spend minimum the work $kT \ln 2$. Or two times that work, because this is a 2-bit-information?

At first sight, Szilard’s theorem seemed to cure the perpetual mobile of second kind. But once we dig into the details, the situation becomes more and more absurd. Given this situation, it seems advisable to look a little bit more skeptical onto Szilard’s alleged perpetual mobile of second kind.

A valid “gedankenexperiment” may assume a setup which is today (and may be forever) well beyond our technical capabilities and available resources. But it must not be in conflict with well-established laws of nature. Szilard’s perpetual mobile fig. 2 does *not* meet this requirement. Szilard’s essential error was, that he neglected thermal fluctuations, which induce van der Waals forces. It’s easy to see that van der Waals forces actually are dominating the machine, while the tiny pressure due to the 1-atom-gas is almost negligible. Let’s explicitly compute the contribution of these two types of forces in process step ④:

At begin of this step, the piston is shifted-in to the middle of the vessel, i. e. $L = L_0$. At the end of this step the piston is pushed fully out, i. e. $L = 0$. During the stroke, the 1-atom-gas exerts onto the piston the force

$$F(L) = p \cdot A \stackrel{(7)}{=} \frac{kTA}{V} = \frac{kTA}{V_m - LA} . \quad (28)$$

To get a high pressure, the temperature must be as high as possible, and the volume as small as possible. Let's assume $T = 1000$ K, $V_m/2 = 1 \mu\text{m}^3$, $A = 1 \mu\text{m}^2$, and $L_0 = 1 \mu\text{m}$.

The attractive van der Waals force between two metal plates of area A and distance D is approximately [11, eq. (36)]

$$F(D) = \frac{\pi^2 \hbar c A}{240 D^4} . \quad (29)$$

The balance of the van der Waals forces, which the right piston and the left vessel wall exert onto the left piston in course of process step ④ is indicated in fig. 3 as a red line, while the force due to the pressure of the 1-atom-gas is indicated by the blue line. Note the logarithmic scale! The effect of van der Waals forces is by many orders of magnitude stronger than the gas pressure, which shall drive Szilard's machine.

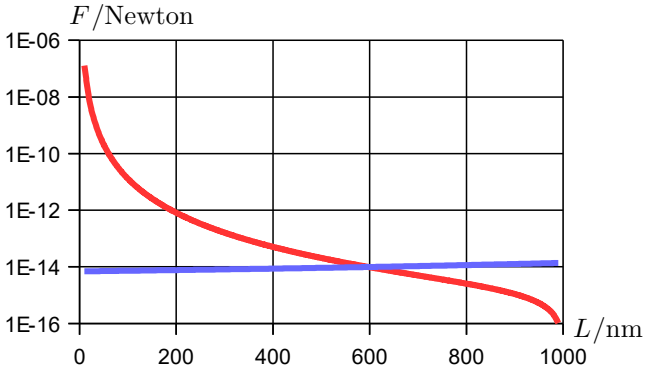


Fig. 3: Force due to pressure of the 1-atom-gas (blue line) and van der Waals force (red line)

To avoid this effect, we could construct the vessel as a very long tube, such that the pistons never come close to the left or right vessel walls. But that would only shift the problem, because then we would get a huge net effect due to van der Waals attraction between the piston and the partition in process step ②. No matter how we construct the details of Szilard's machine, van der Waals forces definitively are not negligible. Just the opposite: Van der Waals forces are dominating, while the tiny pressure due to the single atom is almost negligible.

This fact does not yet disprove Szilard's argument. *If* the van der Waals forces should be 100 % conservative, then we would in course of a full machine cycle get back 100 % the work, which needs to be done in some steps of the cycle against the van der Waals forces.

But that is not to be expected, by the very nature of van der Waals forces. Van der Waals forces are caused by induced dipole moments of atoms and molecules, which again are caused by spontaneously fluctuating dipole moments of other atoms and molecules [12]. Due to the fluctuating van der Waals forces, the machine will in some cycles extract *much* more heat than $Q = kT \ln 2 = (25)$ from the bath, and convert it into usable work. In other cycles, the machine will not convert heat into useful work, but convert work into heat, and dissipate that heat into the bath. The essential thing is, that the effect of the fluctuating van der Waals forces will in any case be larger by several orders of magnitude than the tiny force due to the single rare-gas atom.

Furthermore the appearance of van der Waals forces between the pistons and the partition reminds us, that the van der Waals force between the 1 rare-gas atom, which drives Szilard's machine, and the vessel walls must not be ignored. Due to the induced electric dipole moment, the atom will quite often be adsorbed to the surfaces of vessel, partition, and pistons, and not at all behave

like an ideal gas.

We made a further oversimplification, when we assumed that the gas pressure of the vessel walls and the pistons is zero. Of course the gas pressure of the solid surfaces of many metals is so small, that it is not measurable, not even at 1000 K. But in a consistent description of a machine, whose functionality rests on the tiny pressure of a 1-atom-gas, we certainly must not ignore the metal atoms, which sometimes will detach from the metal surfaces, drift through the vacuum, and demolish the functionality of Szilard's machine.

Note that our refutation of Szilard's alleged perpetuum mobile of second kind follows *exactly* the reasoning as outlined by Smoluchowski [13] already in his well-known 1912 article on the relation of thermal fluctuations and the second law of thermodynamics. In that article Smoluchowski emphasized that in any discussion about proposals for a perpetuum mobile of second kind, which is based onto thermal fluctuations on the scale of atoms and molecules, we must not forget that *all* parts of that machine are subject to such fluctuations. Once all parts of the machine are considered consistently with taking into account thermal fluctuations, the faults of those designs soon become obvious, like the faults in the concept of Szilard's machine.

If we want to build a consistent model with the vessel, the partition, and the pistons as rigid classical bodies, and van der Waals forces being negligible, then we must not fill only one atom into the machine, but a sufficiently large amount of atoms as sketched in fig. 4 on the facing page.

Considering the diagram fig. 3, less than something like $N \approx 10^6$ atoms would hardly be sufficient to make the van der Waals forces negligible. But then it becomes quite unlikely that in process step ①, when the partition is shifted-in, the numbers of atoms in the left and right cell significantly differ. Only in 2 of $2^{10^6} \approx 10^{18}$

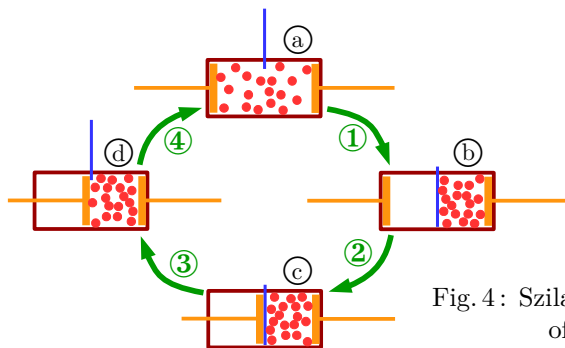


Fig. 4: Szilard’s perpetual motion machine of second kind, corrected

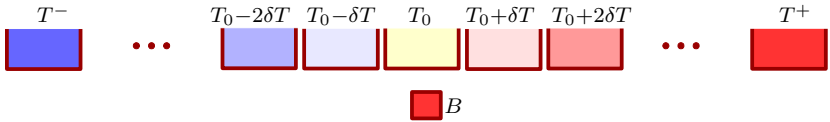
cycles will all 10^6 atoms be trapped by chance in one cell, and the other cell will be empty. If the machine runs at 1 cycle per second, then the mean waiting time for that lucky event is about $18 \cdot 10^9$ years.

Of course we don’t need that perfect event. A cycle with — say — $6 \cdot 10^5$ atoms in one cell and $4 \cdot 10^5$ atoms in the other may still allow for extraction of a little bit of work. The essential thing is that with this type of considerations, the second law turns from a matter of principle to a matter of probability. The second law in this understanding doesn’t say “an isothermal cyclic machine will not extract heat from a bath and convert it into usable work in any cycle”. Instead it says “an isothermal cyclic machine will not extract a measurable amount of heat from a bath and convert it into a measurable amount of usable work in almost all cycles”, and statistical thermodynamics give a precise quantitative value for the rare exceptions.

With the probabilistic interpretation of entropy, there exists absolutely no problem with Szilard’s machine fig. 4. It’s merely unlikely, but not impossible, that the number of atoms trapped by chance in process step ① in one cell differs significantly from the number of atoms trapped in the other cell.

Once we accept that the second law may be violated in rare events, Szilard's perpetual mobile turns into an argument *against* his theorem (16a), due to the absurd consequences at which we arrived when we assumed the theorem to be true.

3.3. The thermal memory



The third device, by which Szilard tried to demonstrate a relation between information and thermodynamic entropy, is an instrument which can measure the property Z of an object. This property can assume only two values: Z^+ or Z^- . If the instrument observes Z^+ , then a body B is first thermally contacted to a heat bath of temperature T_0 , then to a bath of temperature $T_0 + \delta T$, then to a bath of temperature $T_0 + 2\delta T$, and so on, and eventually to a bath of temperature T^+ . When the body B has assumed the temperature T^+ , it is thermally isolated from the baths, and the measurement is completed.

If the instrument observes Z^- , then the body B is first thermally contacted to the heat bath of temperature T_0 , then in infinitesimal steps cooled down to T^- , and then thermally isolated from the baths.

In either case, the result (Z^+ or Z^-) of the measurement is after the measurement documented by the temperature (T^+ or T^-) of the body B . Szilard emphasized that the heating of the body from T_0 to T^+ , and the cooling of the body from T_0 to T^- , is done reversibly due to the small steps δT , while the heating or cooling of the body from the previous result to T_0 is done irreversibly.

Now Szilard entertained some rather intricate considerations,

to demonstrate that this storage of measurement results increases the entropy of the baths by $\Delta S_{\text{bath}} = k \ln 2$ per 1-bit result, as postulated in his theorem (16a). Those considerations are — in my opinion — not worth the effort, because that setup of thermal baths is anyway quite arbitrary. The mechanism could for example be constructed such, that the body B never touches any heat bath, i. e. that merely the position of B codes for the result, with “ B nearby (but never touching) the bath T^+ ” coding for the measured result Z^+ , and “ B nearby the bath T^- ” coding for the measured result Z^- . Then the memory would still serve it’s purpose (storing the result until the next measurement is started), but the entropy change would be zero. Thereby this machine becomes an argument *against* Szilard’s theorem of an unavoidable entropy change upon measurements.

It’s probably fair to say that none of the three machines, which Szilard presented, can be acknowledged a sufficient argument for his theorem (16a). We confirmed an entropy reduction due to the measurement in case of the self-measuring gas. But this entropy reduction resulted from the standard rules of Gibbs’ thermodynamics. It was of no relevance for our computation that the considered process step was a “measurement”. In case of the perpetuum mobile, the assumption of entropy reduction in course of the measurement resulted in absurd consequences. And the thermal memory could easily be modified to an example against Szilard’s theorem (16a).

4. Landauer’s principle

If the input of a logical operation can be uniquely concluded from the output, then that operation is called “logically reversible”. The hallmark of logically reversible operations is a unique one-to-one map between the input configurations and the output configurations. Logical operations, which channel several different

input configurations into one output configuration are logically irreversible. Logically irreversible operations reduce the information entropy, while logically reversible operations don't.⁸

Here are two most simple examples:

INVERT		RESTORE TO ONE	
input	output	input	output
1	0	1	1
0	1	0	1

The INVERT operation is logically reversible. In contrast, from the output 1 of the RESTORE TO ONE operation, we can not conclude whether the input is 1 or 0.

Let's compute the changes $\Delta H = H_{\text{output}} - H_{\text{input}}$ of information entropy generated by these two operations. If we assume for the input data the probabilities $p(0) = p(1) = 1/2$, then we get

$$\begin{aligned}
 H(\text{INVERT, input}) &\stackrel{(15)}{=} -2 \frac{1}{2} \log_2 \frac{1}{2} = 1 \text{ bit} \\
 H(\text{INVERT, output}) &\stackrel{(15)}{=} -2 \frac{1}{2} \log_2 \frac{1}{2} = 1 \text{ bit} \\
 \implies \Delta H(\text{INVERT}) &= 0
 \end{aligned} \tag{30a}$$

$$\begin{aligned}
 H(\text{RESTORE TO ONE, input}) &\stackrel{(15)}{=} -2 \frac{1}{2} \log_2 \frac{1}{2} = 1 \text{ bit} \\
 H(\text{RESTORE TO ONE, output}) &\stackrel{(15)}{=} \frac{1}{1} \log_2 \frac{1}{1} = 0 \\
 \implies \Delta H(\text{RESTORE TO ONE}) &= -1 \text{ bit} .
 \end{aligned} \tag{30b}$$

Rolf Landauer (1927–1999) published in 1961 an influential article [14], in which he asserted what became well-known under the name

⁸ A device which channels one input configuration into several different output configurations, and thereby increases information entropy, would not be a functional logical device but something like a random number generator.

Landauer's principle: If a logical operation reduces the information entropy of the processed data by

$$\Delta H = -1 \text{ bit} , \tag{31a}$$

then the thermodynamic entropy of the hardware, which implements that logical operation, is reduced by minimum

$$\Delta S = -k \ln 2 .$$

According to the second law of thermodynamics, the entropy reduction of the hardware must be compensated by an entropy increase of the environment of minimum same size:

$$\Delta S_{\text{bath}} \geq k \ln 2 \tag{31b}$$

Thus at temperature T minimum the heat²

$$-Q \geq kT \ln 2 \tag{31c}$$

must be dissipated, and consequently according to the first law minimum the work

$$W \geq kT \ln 2 \tag{31d}$$

must be done, to accomplish any logical operation which reduces the information entropy of the processed data by 1 bit.

The similarity of Landauer's principle and Szilard's theorem is striking. A minimum entropy reduction $\Delta S = -k \ln 2$ of the hardware is postulated

- * “per bit of measurement result for any measurement and storage of the result” in case of Szilard's theorem (16a), respectively
- * “per bit of information entropy reduction of the processed data” in case of Landauer's principle (31a).

As a measurement clearly is correlated with a reduction of information entropy (we get the information of the — previously unknown — actual value of a physical quantity), the theorems of Szilard and Landauer are essentially identical. The only difference is that Landauer had the precise notion “information entropy” (defined by Shannon in 1948) available, while Szilard relied on the somewhat vague notion “measurement”. Obviously Szilard’s prior work was unknown to Landauer, as he did not cite it in his 1961 publication.

With regard to logically reversible operations, Landauer asserted in later publications, that

“**reversible classical computation** [...] can be accomplished with as little energy dissipation, per step, as desired.” [15] (32)

Thus — in his opinion — the entropy increase $\Delta S_{\text{bath}} \geq k \ln 2$ per bit of information entropy reduction is inevitable only in logically irreversible operations.

Whether lossless (reversible or irreversible) classical computation is possible in principle, is a quite difficult and intricate question. Lossless computation has been advocated in particular by Bennet [16]. Objections have been raised e. g. by Porod et. al. [17]. The question is controversial still by today.⁹ I will in this article concentrate on Landauer’s principle (31), and touch the issue (32) of loss-less computation only sporadically.

To evaluate Landauer’s principle (31) in more detail, it is instructive to run the process cycle of (the amended version of)

⁹ A “commercial” argument: High-power processors are a market of about 10^{11} US\$ per year, and the power dissipation of the devices is a predominant problem, severely impeding the progress of computer technology. Why, then, did no player in this market make — despite the huge 10^{11} US\$ incentive — the slightest progress towards loss-less computation, even though the idea for loss-less computation is available since more than half a century?

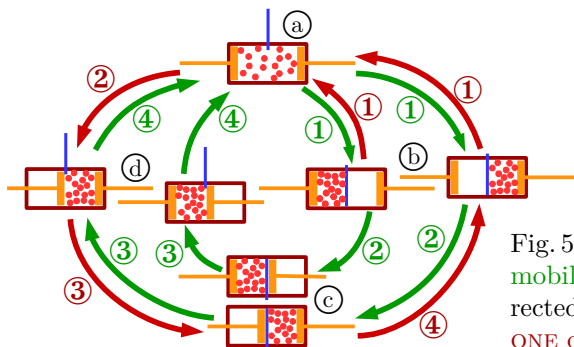
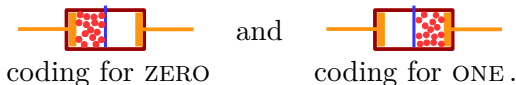


Fig. 5: Szilard's **perpetuum mobile of second kind corrected**, and the **RESTORE TO ONE** operation.

Szilard's perpetuum mobile of second kind (sketched in fig. 5) in counter-clockwise direction. Remember in the following discussion, that thermal fluctuations and van-der-Waals forces are negligible, because the machine is not run with only 1 atom, but with a sufficiently large number N of atoms.¹⁰

The machine is interpreted as a binary memory with



- ⓑ It's of no relevance whether the memory is in state ZERO (as sketched in the inner circle of fig. 5), or in state ONE (as sketched in the outer circle). Under the assumption¹¹ that either of the both states is realized with probability $1/2$, the

¹⁰ Thermal fluctuations and van-der-Waals forces are negligible, because the machine is not run with only 1 atom, but with a sufficiently large number N of atoms. Therefore the discussion of the machine fig. 5 will not help to assess whether loss-less computation is possible or not, because the objections against that assumption concentrate exactly on the issue of thermal fluctuations.

¹¹ Landauer emphasizes that this assumption results into maximum information entropy of the input state. In many realistic cases, the input information entropy may be lower.

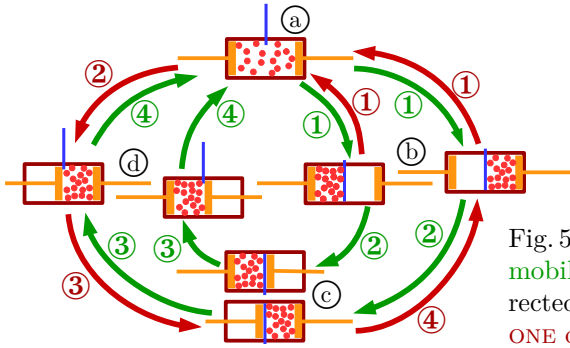


Fig. 5: Szilard's **perpetuum mobile of second kind corrected**, and the **RESTORE TO ONE** operation.

information entropy of state (b) is

$$H \stackrel{(15)}{=} \overbrace{-\frac{1}{2} \log_2 \frac{1}{2}}^{\text{ZERO}} \overbrace{-\frac{1}{2} \log_2 \frac{1}{2}}^{\text{ONE}} = \log_2 2 = 1 \text{ bit} \quad (33)$$

- ① In the first step of the RESTORE TO ONE process the partition is pulled out. From the point of view of thermodynamics, this is just the experiment of Gay-Lussac: The mean kinetic energy

$$U = \frac{3NkT}{2} \quad (34)$$

of the N atoms does not change. And the gas does no work. Consequently, according to the first law of thermodynamics

$$dU \stackrel{(1a)}{=} \delta Q + \underbrace{dW}_0 = 0 \implies \delta Q = 0 ,$$

i. e. there is no heat exchange with the bath. But the entropy of the gas does increase (remember that the relation $T\Delta S = \delta Q$ only holds for reversible processes, while ① is an irreversible

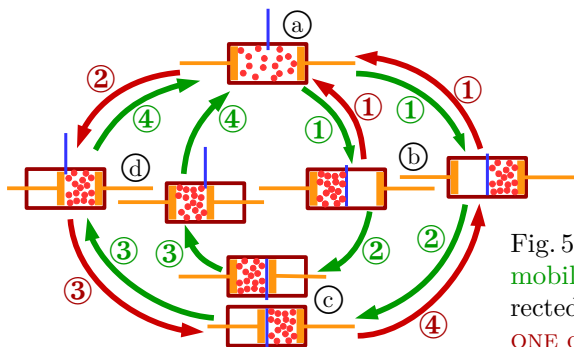


Fig. 5: Szilard's **perpetuum mobile of second kind corrected**, and the **RESTORE TO ONE** operation.

process step). As the volume accessible to the N -atom-gas doubles, it's thermodynamic entropy increases by

$$\Delta S = Nk \ln 2 . \quad (35)$$

Nothing sensible can be said about the information entropy of the intermediate state (a), because this state codes neither for ZERO nor for ONE.¹²

- (2) In the second step of the RESTORE TO ONE process, the gas is shifted by means of the left piston into the right half of the vessel, which codes for ONE. Thereby the work²

$$W = - \int_{V_m}^{V_m/2} dV p(V) = -NkT \int_{V_m}^{V_m/2} \frac{dV}{V} = NkT \ln 2 \quad (36)$$

must be done, and the entropy of the gas is decreased by

¹² Some readers might be tempted to argue that the data, which were stored by the device in state (b), have been “deleted” or “erased” due to step (1). Landauer's principle, however, is not about the erasure of data, but about the reduction of information entropy. The RESTORE TO ONE operation reduces the information entropy of data to zero. But even data with information entropy zero are data, while in the undefined intermediate state (a) no data exist, to which any information entropy could be assigned.

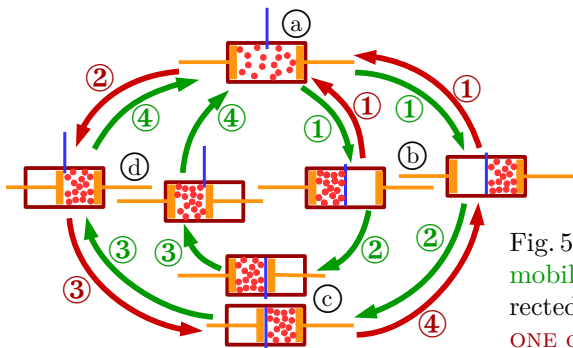


Fig. 5: Szilard's **perpetuum mobile of second kind corrected**, and the **RESTORE TO ONE** operation.

$$\Delta S = -Nk \ln 2 . \quad (37)$$

As this is a reversible process step, the heat²

$$-Q = -T\Delta S = NkT \ln 2 \quad (38)$$

is transferred to the bath, increasing the entropy of the bath by

$$\Delta S_{\text{bath}} = -\Delta S = +Nk \ln 2 . \quad (39)$$

The device now is with probability 1 in state ONE, and with probability 0 in state ZERO. Hence the information entropy of state (d) is

$$H \stackrel{(15)}{=} \underbrace{-1 \log_2 1}_0 \underbrace{-0 \log_2 0}_0 = 0 . \quad (40)$$

- ③ The partition is shifted into the vessel, to make sure that the ONE state will be conserved, even if the force, which pushed the left piston in, is relaxed. The partition, sliding in friction-less along the piston, can be inserted with no expense of energy.¹⁰

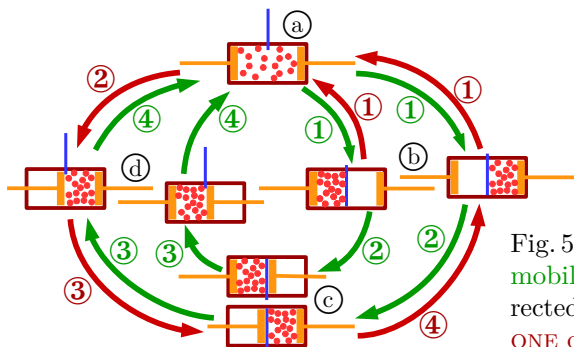


Fig. 5: Szilard's **perpetuum mobile of second kind corrected**, and the **RESTORE TO ONE** operation.

Furthermore neither thermodynamic entropy nor information entropy are changed.

- ④ To complete the cycle, the left piston is pulled out. There is no change of energy¹⁰ nor any change of thermodynamic entropy or information entropy in this last step of the RESTORE TO ONE process.

All exchange of work and heat, and all changes of information entropy and thermodynamic entropy happen during steps ① and ②. The thermodynamic entropy changes (35) and (38) of the gas cancel. In total there is a decrease $\Delta H = -1$ bit of information entropy and an increase $\Delta S_{\text{bath}} \stackrel{(39)}{=} Nk \ln 2$ of thermodynamic entropy. Furthermore the work $W \stackrel{(36)}{=} NkT \ln 2$ is done and the heat $Q \stackrel{(38)}{=} -NkT \ln 2$ is dissipated² to the bath in the RESTORE TO ONE process implemented by the device of figure 5. In the next section, realistic examples for devices with $N = 1$ (which of course significantly differ from Szilard's original 1-atom-machine) will be presented. Thus all above results match with Landauer's principle (31).

It is easy to show, however, that the heat dissipation in course of the (logically irreversible!) RESTORE TO ONE operation can be reduced to zero¹⁰, if the irreversible expansion of the gas in

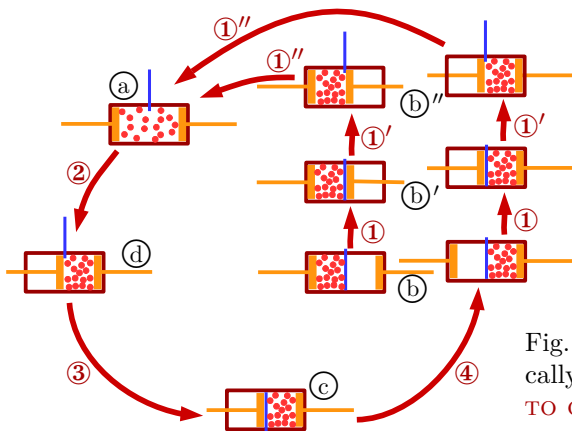


Fig. 6: A thermodynamically reversible **RESTORE TO ONE** operation

the first process step is replaced by a reversible expansion. This thermodynamically reversible process is displayed in figure 6. The first process step is now sub-divided into the three steps ①, ①', and ①''.

In step ① the piston is shifted into the empty cell. In the discussion of Szilard's perpetuum mobile of second kind in section 3.2, we found convincing reasons for the assumption, that no energy is needed¹⁰ for this step, and that there is no change of entropy.

In step ①' the partition is pulled out, and in step ①'' the gas does the work² $-W = NkT \ln 2$ during the reversible expansion. Thereby the thermodynamic entropy of the gas is increased by $\Delta S = Nk \ln 2$, and it absorbs from the bath the heat $Q = T\Delta S = NkT \ln 2$, while at the same time the entropy of the bath is decreased by $\Delta S_{\text{bath}} = -\Delta S = -Nk \ln 2$.

The other process steps ②, ③, ④ are done like in the thermodynamically irreversible process fig. 5. As the work done by the gas and the heat absorbed from the bath in step ①'' exactly compensate the work needed and the energy dissipated in step ②, now for the total RESTORE TO ONE operation we still have the reduction

$\Delta H = -1$ bit of information entropy, but no net work¹⁰ needs to be done, and no net heat is dissipated.

Thus, if thermal fluctuations would be negligible, then the logically irreversible RESTORE TO ONE operation could be implemented with zero power dissipation, in contradiction to Landauer’s principle. This proves the conclusion:

IF there should exist an inevitable minimum power dissipation > 0 for the physical implementation of a logical operation, THEN that minimum power dissipation is caused by inevitable thermal fluctuations, which of course would affect both logically reversible and logically irreversible operations alike. Therefore, if a minimum power dissipation of logical operations should exist, it is — in contradiction to Landauer’s principle (31a) — not correlated with the logical reversibility or irreversibility of that operation. (41)

How came Landauer’s error about? Like Szilard, Landauer didn’t state his theorem explicitly. Instead (31a) is a condensed form of arguments presented in section “4. Logical irreversibility and entropy generation” of his article [14]. In this section Landauer considers the example of the RESTORE TO ONE operation, and argues:

“Note that our argument here does not necessarily depend upon connections, frequently made in other writings, between entropy and information. We simply think of each bit as being located in a physical system, with perhaps a great many degrees of freedom, in addition to the relevant one. However, for each possible physical state which will be interpreted as a ZERO, there is a very similar possible physical state in which the physical system represents a ONE. [...]

Consider a statistical ensemble of bits in thermal equilibrium¹³. If these are [in a RESTORE TO ONE operation] all reset to ONE, the number of states covered in the ensemble has been cut in half. The entropy therefore has been reduced by $k \log_e 2 = 0.6931 k$ per bit. The entropy of a closed system, e. g., a computer with its own batteries, cannot decrease; hence this entropy must appear elsewhere as a heating effect, supplying $0.6931 kT$ per restored bit to the surroundings. This is, of course, a minimum heating effect, and our method of reasoning gives no guarantee that this minimum is in fact achievable.”

At first sight these arguments seem highly plausible¹⁴ and almost self-evident, because they are presented as an immediate consequence of the second law. But actually a quite trivial error has crept into Landauer's considerations.

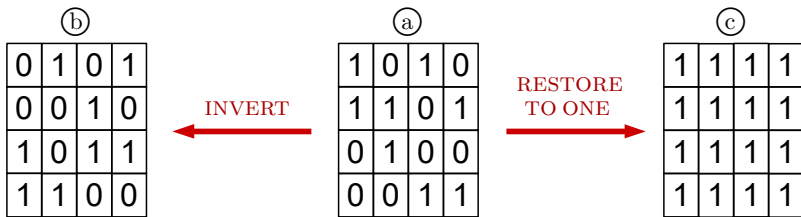


Fig. 7: A 16 bit memory

Consider the 16-bit memory displayed in fig. 7. In state (a), the memory stores ONE in 8 cells, and ZERO in the other 8 cells. Due

¹³ From the context it is obvious, that Landauer is speaking of an ensemble of bits, in which ONE and ZERO are showing up with approximately same frequency.

¹⁴ At least the present author needed some time and considerations, until he detected Landauer's error. Maybe the reader is smarter, and will figure out the issue immediately?

to a RESTORE TO ONE operation, all 16 cells are transformed to ONE, and thereby — according to Landauer’s reasoning — “the number of states covered in the ensemble has been cut in half. The entropy therefore has been reduced by $k \log_e 2 = 0.6931 k$ per bit.”

No! This is not correct. Landauer overlooked that the data (ONES and ZEROS stored in a memory or handled in course of a running computer program) are elements of an *ordered* set, but not elements of an *disordered* set.



Fig. 8: A gas of 16 atoms

To understand the argument, consider the (almost ideal) gas of 16 rare-gas atoms, displayed in fig. 8. The sketches (a) and (c) show the states of the gas immediately before the blue partition was shifted in, and (b) and (d) are the states of the gas immediately after the blue partition was shifted in.

The probability, that by chance n atoms spontaneously assemble in the left half of the vessel immediately before the partition is shifted in, is

$$P_{n \text{ left}} = \frac{1}{2^{16}} \binom{16}{n}, \quad n = 0, 1, 2, \dots, 16. \quad (42)$$

Thus the probability to end with states (b) or (d) of fig. 8, respectively, is

$$P_{\textcircled{b}} = \frac{1}{2^{16}} \binom{16}{8} \approx 0.196, \quad P_{\textcircled{d}} = \frac{1}{2^{16}} \approx 1.53 \cdot 10^{-5}$$

$$\Rightarrow \frac{P_{\textcircled{b}}}{P_{\textcircled{d}}} = \binom{16}{8} = 12870 \quad (43)$$



Fig. 8: A gas of 16 atoms

The larger probability (43) of state (b) versus (d) correlates with a larger thermodynamic entropy. By means of Boltzmann's formula (11), we get

$$S_{\text{(b)}} - S_{\text{(d)}} = k \ln \frac{\Phi_{\text{(b)}}}{\Phi_{\text{(d)}}} = k \ln \left(\frac{V}{V/2} \right)^{16} = 16 k \ln 2, \quad (44)$$

because the phase space covered by the ensemble of the 16 atoms is twice as large in states (a) and (b) than in states (c) and (d).

Let's interpret an atom in the left half of the vessel fig. 8 as representing a ZERO, and an atom in the right half of the vessel as representing a ONE. If the RESTORE TO ONE operation is applied to the "ensemble in thermal equilibrium", then this operation is represented by the transition from fig. 8 (b) to fig. 8 (d), and indeed — as Landauer wrote — the thermodynamic "entropy therefore has been reduced by $k \log_e 2 = 0.6931 k$ per bit." This is obviously the scenario envisaged by Landauer.

But the rare gas atoms in a vessel differ basically from the bits in a computation, because the atoms are a *disordered* set, while the bits in a computation are an *ordered* set. To see the impact of order versus disorder of the two sets, we start with an empty vessel (perfect vacuum), and fill the vessel fig. 8 and the memory fig. 7 by tossing a coin:

If we get a head, we store a ONE into the first cell of the memory, and put an atom into the right vessel compartment. In case of a tail, we store a ZERO into the first cell of the memory, and put an atom into the left vessel compartment.

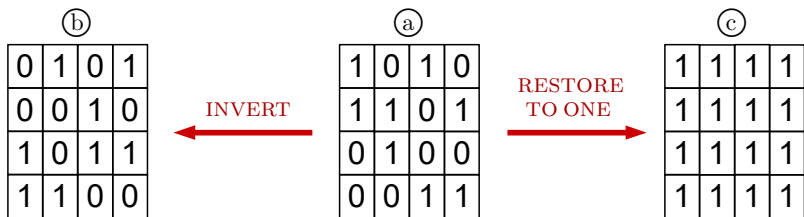


Fig. 7: A 16 bit memory

This procedure is repeated for the second memory cell, then for the third memory cell, and so on. The probability that we get the result fig. 7 (a) by this method is 2^{-16} . The probability that we get the result fig. 7 (c) as well is 2^{-16} . In contrast, the probability that we get the result fig. 8 (d) is 2^{-16} , while the probability of the result fig. 8 (b) is $12870 \cdot 2^{-16}$, see (43).

Thus the RESTORE TO ONE operation does indeed reduce the probability (and hence the thermodynamic entropy) of the state of the 16-atom-gas, but it does *not* reduce the probability (and hence the thermodynamic entropy) of the memory state. The difference comes about, because the sequence, by which the atoms are filled into the vessel compartments, is irrelevant, while in the operations of computers, the exact cell address of any datum in the memory *is* relevant.

The same holds true if we not consider the hardware state of a memory, but the hardware state of a logical device which is handling a single datum being processed in course of a running computer program. It is not correct to say that the thermodynamic entropy of the input of a RESTORE TO ONE device is higher than the thermodynamic entropy of the output, because the state of the input hardware sometimes represents a ONE and sometimes represents a ZERO, while it's output state always represents a ONE. For the entropy of the hardware state representing a ONE or a

ZERO at some port of some logical device in some certain step of the program, it is of no relevance whether or not in some other step of the same program the same hardware is representing a ONE or a ZERO, like it is of no relevance for the entropy of a memory hardware representing a ONE or a ZERO, whether or not the hardware of some other cell of the same memory is representing a ONE or a ZERO.

Thus, once we understand that the data handled by a computer are elements of an *ordered* set, Landauer's argument for a correlation between the reduction of information entropy of the data, and an alleged reduction of thermodynamic entropy of the hardware which is handling the data, cited on page 45, is completely demolished.

5. Experimental evaluations of the Landauer limit

The argument, cited on page 45, by which Landauer wanted to prove his principle (31), is clearly wrong. This does not exclude, however, that there indeed might exist — as stated in (41) — a minimum power dissipation for the physical implementations of logical operations, and that its value might be $kT \ln 2$.

This hypothetical minimum power dissipation often is called the “Landauer limit”. It should better be called Brillouin limit, however, because it was not Landauer but L. Brillouin, who thoroughly discussed this limit in his 1953 article [18]. The energy to store a measurement result must be by order of magnitude $\gtrsim kT$, to make it discernible from a mere thermal fluctuation. Brillouin asserted¹⁵ that minimum $kT \ln 2$ must be spent per bit, to make stored data discernible with just a 50 % probability from thermal fluctuations.

The hardware considered in figures 5 and 6 is of course not

¹⁵ I will not assess whether his proof [18] is valid and correct.

suites to approach that limit, because Szilard's machine can be realized only with a huge number N of atoms. Only since 2012 have experimentalists been able to approach the limit

$$\Delta S = -k \ln 2 \quad (45a)$$

with 1-particle-devices which are realizing logical 1-bit-processes, and measure (respectively conclude from other experimental parameters) the related tiny work

$$W = kT \ln 2 \approx 3 \cdot 10^{-21} \text{J} \quad \text{at room temperature ,} \quad (45b)$$

which must be done to drive the process.

5.1. A bead trapped in the laser focus

Berut et. al. [19] were the first to report a successful experiment of that type. The title "Experimental verification of Landauer's principle linking information and thermodynamics" of their publication, however, is misleading. Their experiment did not demonstrate whatever link between information and thermodynamics, and in particular it did not verify Landauer's principle (31). Instead their experiment demonstrated that the work, which had to be done to drive the thermodynamically irreversible process, which they analyzed, approached from above the limit $kT \ln 2$.

In their experiment, a silica bead of $2 \mu\text{m}$ diameter, which was suspended in water, was trapped in the focus of a laser beam. The laser beam was focused onto two nearby points with a switching rate of 10 kHz, such that a double-well potential as sketched in fig. 9 on the next page was created, with distance $0.9 \mu\text{m}$ between the two minima, and a barrier height of about $2.2 kT$ between the two wells.

At the begin of each experimental run, the bead was in one well, the other well was empty. Note that the bead diameter was

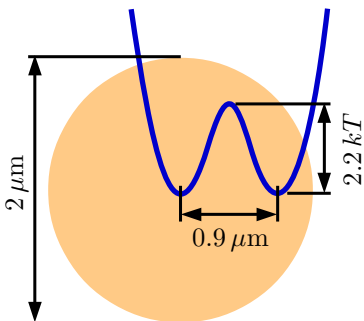


Fig. 9: The silica bead in the double-well potential

much larger than the distance of the minima of the two wells. The position of the bead could still be clearly assigned to one of the wells, because it was observed at 502 Hz with a fast camera, which allowed (after picture treatment) for a resolution of better than 10 nm .

Then the water cell was moved for a time τ with speed v relative to the laser focus. The speed was during the time interval τ linearly increased from $v = 0$ to $v = v_{\max}$. Due to the viscous drag of the water, the bead felt the force

$$F = \gamma \cdot v \quad (46a)$$

$$\gamma = 1.89 \cdot 10^{-10} \text{ Ns/m} = \text{coefficient of friction} ,$$

which was driving the bead towards the empty well. This process can of course be interpreted as a realization of the logically irreversible RESTORE TO ONE operation, with “bead in the left well” coding for ZERO, and “bead in the right well” coding for ONE. Note, however, that this process can as well be interpreted as a realization of the logically reversible INVERT operation.

The time τ was varied in the range from 5 s to 40 s, and the work

$$W(\tau) = \int_0^{\tau} dt F(t) v(t) \quad (46b)$$

done onto the bead was computed. It turned out that $W(\tau)$ is an exponentially decreasing function, which does not converge against zero, but against

$$\lim_{\tau \rightarrow \infty} W(\tau) = kT \ln 2 . \quad (46c)$$

5.2. A particle in the electrokinetic trap

Jun et. al. [20] trapped a 200 nm fluorescent particle not in the focus of a laser, but in the electric potential created by a “electrokinetic feedback trap”. This is a trap, in which a varying voltage is applied to electrodes. Depending on the observed position of the bead, the voltage was chosen such that the bead felt a double-well potential with variable height of the intermediate barrier. The update time of the feedback loop was 10 ms .

Two types of processes were run: In process ①, the bead was shifted (due to frictional forces of the water) from that well, where it was first observed, to the other well, thereby simulating the logically irreversible RESTORE TO ONE operation, or the logically reversible INVERT operation. In process ②, the intermediate barrier was lowered, and after some time raised again, without exerting frictional forces due to the water. In this process, the bead ended up in either well with probability $1/2$. This operation may be called RANDOMIZE. It is a defining requirement of any logical operation, that the output data are uniquely determined by the input data. The RANDOMIZE operation does not meet this requirement, hence it is no logical operation at all.

Again it turned out that the (computed) work, which was done on the particle, was lower, if the process was run slower. The

extrapolated computed work done on the particle was

$$\lim_{\tau \rightarrow \infty} W(\tau) = \begin{cases} kT \ln 2 & \text{with process ①} \\ 0 & \text{with process ②.} \end{cases} \quad (47)$$

Shifting a particle through a viscous fluid is by construction a thermodynamically irreversible process. Jun et. al. demonstrated, that a logical operation, if realized by a thermodynamically irreversible process, needs minimum the work $kT \ln 2$ per processed bit of information.

Actually Jun et. al. interpreted their results as support for Landauer's principle (31), because no energy was dissipated in process ②. But this interpretation is clearly mistaken:

In process ①, work was done to overcome the viscous friction and shift the bead into the target well. In process ②, the experimenters simply leaned back, and let the bead drift into whatever well it wanted to drift. Thus quite trivially work had to be done in process ①, and no work needed to be done in process ②.

Process ②, which gave the $\lim_{\tau \rightarrow \infty} W(\tau) = 0$ result, is realizing the RANDOMIZE operation, which is no logical operation. Process ①, which gave the $\lim_{\tau \rightarrow \infty} W(\tau) = kT \ln 2$ result, is the (thermodynamically irreversible) realization of both the logically reversible INVERT operation and the logically irreversible RESTORE TO ONE operation. Hence Jun et. al. did *not* demonstrate that a logical operation (be it logically reversible or irreversible) can be performed with no energy dissipation.

5.3. Switching a nanomagnetic memory

A further experiment, which deserves mention, has in 2016 been published by Hong. et. al. [21]. These experimenters measured the (magnetic) work needed to switch 1-domain nanomagnetic memories of lateral dimensions slightly below 100 nm. The nanomagnets







had an “easy” axis and a “hard axis”. More energy is needed to align the magnetic moment to the hard axis than to the easy axis. Hence in thermal equilibrium the magnetic moment is aligned either parallel (this codes for ZERO) or antiparallel (this codes for ONE) to the easy axis.









To switch the memory from ZERO to ONE or from ONE to ZERO, work must be done due to externally applied magnetic fields. Hong. et. al. measured the work needed to switch the memory, and found values which come close to $W \approx kT \ln 2$, but scattered by 50 % to 100 % around this value.

Hong. et. al. interpreted this result as a confirmation of the Landauer limit $W \geq kT \ln 2$. As their process, different from the bead shifted against water friction, is not obviously dissipative, a different speculation is tempting: Of course we must do work to turn the magnetic moment from the easy axis into the hard axis. But why don't we get this work 100 % back, when the moment is relaxed from the hard axis to it's new orientation along the easy axis? Indeed the many results with W well below $kT \ln 2$ might indicate (??) that this process might ideally be lossless (i. e. thermodynamically reversible), and that the results $W > 0$ might be caused by experimental imperfections. If, on the other hand, the result $W \geq kT \ln 2$ should be confirmed for this system, then this power dissipation must be caused by thermal fluctuations, but not by ordinary friction.






Thus it is a pity that the results of just this experiment are relatively unclear, and do not really confirm or disprove the importance of thermal fluctuations.

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