Fourier Series and Fourier Transform

This is a collection of formulas with minimal explanatory text, intended to assist in the application of the complex Fourier series, the real Fourier series, the discrete complex Fourier series, and the complex Fourier transform.

1. The complex Fourier series

Every complex-valued function $A(t) \in \mathbb{C}$ of a real variable $t \in \mathbb{R}$ can be decomposed into a Fourier series if it is piecewise continuous and square integrable,[1] and if it is either periodic with period T or defined only on the finite interval $[t_0, t_0 + T] \in \mathbb{R}$, so that it can be continued periodically with period T:

$$A(t) = \sum_{k=-\infty}^{+\infty} c_k e^{+ik\frac{2\pi}{T}\cdot t} \quad \text{with } k = 0,\ \pm 1,\ \pm 2,\ \ldots \tag{1a}$$

The complex Fourier coefficients $c_k \in \mathbb{C}$ are

$$c_k = \frac{1}{T}\int_{t_0}^{t_0+T} \mathrm{d}\,t\ A(t) e^{-ik\frac{2\pi}{T}\cdot t} = \frac{1}{T}\int_{-T/2}^{+T/2} \mathrm{d}\,t\ A(t) e^{-ik\frac{2\pi}{T}\cdot t} \quad . \tag{1b}$$

Renaming $c_{-k} \leftrightarrow c_k$ results in these formulas:

$$A(t) \overset{(1a)}{=} \sum_{k=-\infty}^{+\infty} c_k e^{-ik\frac{2\pi}{T}\cdot t} \quad \text{with } k = 0,\ \pm 1,\ \pm 2,\ \ldots$$

[1] All functions that occur in physics satisfy this criterion.

$$c_k \overset{(1b)}{=} \frac{1}{T} \int\limits_{-T/2}^{+T/2} \mathrm{d}\,t\; A(t) e^{+ik\frac{2\pi}{T}\cdot t}$$

So it doesn't matter whether you write $+i$ in the first exponent and $-i$ in the second exponent, or vice versa. Both conventions appear frequently in the literature. You just have to make sure that you always stick to one convention consistently.

2. The real Fourier series

If $A(t)$ is real-valued, then

$$c_0 \overset{(1b)}{=} \frac{1}{T} \int\limits_{-T/2}^{+T/2} \mathrm{d}\,t\; A(t)$$

is clearly real. All other Fourier coefficients $c_k =$ (1b), however, are generally complex even for $A(t) \in \mathbb{R}$. In order to formulate the series expansion with real Fourier coefficients in this case, we use the fact that the Fourier coefficient c_k is obviously complex conjugate to the coefficient c_{-k} in the case $A(t) \in \mathbb{R}$:

$$\frac{1}{T} \int\limits_{-T/2}^{+T/2} \mathrm{d}\,t\; A(t) e^{-ik\frac{2\pi}{T}\cdot t} \overset{(1b)}{=} c_k = \overline{c_{-k}} = \frac{1}{T} \int\limits_{-T/2}^{+T/2} \mathrm{d}\,t\; \overline{A(t) e^{+ik\frac{2\pi}{T}\cdot t}} \tag{2}$$

So, if $A(t) \in \mathbb{R}$, the series (1a) can be written as follows:

$$\begin{aligned} A(t) &\overset{(1a)}{=} \sum_{k=-\infty}^{+\infty} c_k e^{+ik\frac{2\pi}{T}\cdot t} \quad \text{with } k = 0,\, \pm 1,\, \pm 2,\, \ldots \\ &= c_0 + \sum_{k=+1}^{+\infty} \Big(c_k e^{+ik\frac{2\pi}{T}\cdot t} + \overline{c_k} e^{-ik\frac{2\pi}{T}\cdot t} \Big) \quad \text{with } k = 1,\, 2,\, 3,\, \ldots \end{aligned} \tag{3}$$

Using the definitions

$$\frac{a_k - ib_k}{2} := c_k \quad \text{with } a_k,\, b_k \in \mathbb{R} \tag{4a}$$

$$x := k\frac{2\pi}{T} \cdot t \tag{4b}$$

and Euler's formula

$$e^{\pm ix} = \cos x \pm i \sin x \quad , \tag{4c}$$

the Fourier series becomes

$$\begin{aligned} A(t) \overset{(3)}{=} \frac{a_0}{2} + \sum_{k=1}^{+\infty} \Big[\frac{a_k - ib_k}{2} (\cos x + i \sin x) + \\ + \frac{a_k + ib_k}{2} (\cos x - i \sin x) \Big] \\ = \frac{a_0}{2} + \sum_{k=1}^{+\infty} \Big[a_k \cos(k\frac{2\pi}{T} \cdot t) + b_k \sin(k\frac{2\pi}{T} \cdot t) \Big] \end{aligned} \tag{5a}$$

The Fourier coefficients are

$$\frac{a_k - ib_k}{2} \overset{(4a)}{=} c_k \overset{(1b)}{=} \frac{1}{T} \int_{-T/2}^{+T/2} \mathrm{d}\,t\ A(t) e^{-ik\frac{2\pi}{T} \cdot t} \quad .$$

With (4c), this results in the real Fourier coefficients

$$a_k = \frac{2}{T} \int_{-T/2}^{+T/2} \mathrm{d}\,t\ A(t) \cos(k\frac{2\pi}{T} \cdot t) \tag{5b}$$

$$b_k = \frac{2}{T} \int_{-T/2}^{+T/2} \mathrm{d}\,t\ A(t) \sin(k\frac{2\pi}{T} \cdot t) \quad . \tag{5c}$$

3. The discrete complex Fourier series

In many technical applications, $A(t)$ is not a continuous function, but rather a sequence of N discrete values A_n with $n = 0, 1, 2, \ldots, (N-1)$.[2] Then the Fourier coefficients are not determined by the integral (1b), but by the sum

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} A_n e^{-ik\frac{2\pi}{N}\cdot n} \quad , \tag{6a}$$

and the Fourier series (1a) becomes

$$A_n = \sum_{k=0}^{N-1} c_k e^{+ik\frac{2\pi}{N}\cdot n} \quad . \tag{6b}$$

For the continuous series (1) and (5), one must calculate an infinite number of Fourier coefficients (i. e., calculate the sequences analytically, but not numerically) in order to represent the function exactly. For an exact representation of the N elements of the sequence $\{A_n\}$, on the other hand, exactly N Fourier coefficients are required. This is reasonable because the sequences $\{A_n\}$ and $\{c_k\}$ contain exactly the same amount of information, namely $2N$ real numbers each, N for the real parts and N for the imaginary parts of each sequence.

If all $A_n \in \mathbb{R}$ are real, the information content of the sequence $\{A_n\}$ consists of only N real numbers. In the sequence of Fourier coefficients $\{c_k\}$, two imaginary parts vanish due to

$$c_0 \overset{(6a)}{=} \frac{1}{N}\sum_{n=0}^{N-1} A_n \in \mathbb{R} \quad ; \quad c_{N/2} \overset{(6a)}{=} \frac{1}{N}\sum_{n=0}^{N-1} A_n \cdot (-1)^n \in \mathbb{R} \ . \tag{7}$$

[2] This is obviously always the case if the series is not calculated analytically, but on a computer using a finite number of sampling points.

The imaginary parts of all other $N-2$ Fourier coefficients are generally different from zero, however, even in the case of real A_n. But due to $c_{N-k}=\overline{c_k}$ these $N-2$ coefficients contain only $N/2-1$ independent real parts and $N/2-1$ independent imaginary parts:

$$\begin{aligned} c_{N-k} &\overset{(6a)}{=} \frac{1}{N}\sum_{n=0}^{N-1} A_n e^{-i(N-k)\frac{2\pi}{N}\cdot n} \\ &= \frac{1}{N}\sum_{n=0}^{N-1} A_n \underbrace{e^{-iN\frac{2\pi}{N}\cdot n}}_{=1} e^{+ik\frac{2\pi}{N}\cdot n} = \overline{c_k} \end{aligned} \tag{8}$$

Therefore, in the case of real A_n, in addition to c_0 and $c_{N/2}$, only $N/2-1$ further Fourier coefficients c_k need to be calculated; the rest results from (8).

The information content of the sequence $\{c_k\}$ in the case of a purely real sequence $\{A_n\}$ is 2 real parts according to (7) plus $N/2-1$ real parts plus $N/2-1$ imaginary parts according to (8), making a total of N real numbers. Thus the sequences $\{c_k\}$ and $\{A_n\}$ again have the same information content also in this case.

4. The complex Fourier transform

If the complex-valued function $A(t)\in\mathbb{C}$ of a real variable $t\in\mathbb{R}$ is piecewise continuous and square integrable,[3] but — unlike in section 1, $A(t)$ is not periodic and is defined for any $t\in\mathbb{R}$, then $A(t)$ cannot be expanded into the Fourier series (1), but it can be represented by this Fourier transform:

[3] All functions that occur in physics satisfy this criterion.

$$A(t) = \int_{f=-\infty}^{+\infty} \mathrm{d}f \; C(f)\, e^{+if2\pi t} \tag{9a}$$

$$C(f) = \int_{t=-\infty}^{+\infty} \mathrm{d}t \; A(t)\, e^{-if2\pi t} \tag{9b}$$

$$\text{with } t \in \mathbb{R},\ f \in \mathbb{R},\ A(t) \in \mathbb{C},\ C(f) \in \mathbb{C}$$

The Fourier transform (9) can be made plausible by considering the non-periodic function $A(t) \in \mathbb{C}$ as a periodic function with an infinitely large period length $T \to \infty$. Under this condition, in the exponential functions of the Fourier series

$$A(t) \overset{(1a)}{=} \sum_{k=-\infty}^{+\infty} c_k e^{+ik\frac{2\pi}{T}\cdot t} \tag{1a}$$

$$c_k \overset{(1b)}{=} \frac{1}{T} \int_{-T/2}^{+T/2} \mathrm{d}\,t \; A(t) e^{-ik\frac{2\pi}{T}\cdot t} \tag{1b}$$

the step size $(k+1)/T - k/T = 1/T$ becomes infinitely small, so that the transition to continuous variables

$$f := \lim_{T\to\infty} \frac{k}{T} \quad , \quad -\infty \le f \le +\infty \tag{10a}$$

$$\text{step size of } f = \mathrm{d}f = \lim_{T\to\infty} \frac{1}{T} \tag{10b}$$

makes sense. Here, I have emphasized that f can take on any values from $f = -\infty$ to $f = +\infty$. This is because, for any large but finite T, the exponential function takes on the complete spectrum of values from $e^{-i\infty}$ to $e^{+i\infty}$. So it is plausible that it still does so when $T \to \infty$, and that the discrete spectrum simply becomes the continuous spectrum $e^{-i\infty}$ to $e^{+i\infty}$.

Because $c_k \xrightarrow{T\to\infty} 0$, instead of c_k the function

$$C(f) := \lim_{T\to\infty} T \cdot c_k \tag{11}$$

is defined. This function becomes

$$C(f) \overset{(1b)}{=} \lim_{T\to\infty} T \cdot \frac{1}{T} \int\limits_{-T/2}^{+T/2} \mathrm{d}\,t\ A(t) e^{-ik\frac{2\pi}{T}\cdot t} =$$

$$\overset{(10a)}{=} \int\limits_{t=-\infty}^{+\infty} \mathrm{d}t\ A(t)\, e^{-if2\pi t} = (9b)\ .$$

Now we check whether the Fourier series (1a) converges to the Fourier transform (9a) as $T \to \infty$:

$$\lim_{T\to\infty} (1a) = \lim_{T\to\infty} \sum_{k=-\infty}^{+\infty} c_k e^{+ik\frac{2\pi}{T}\cdot t}$$

$$\overset{(10a),(11)}{=} \lim_{T\to\infty} \sum_{k=-\infty}^{+\infty} \frac{1}{T}\, C(f)\, e^{+if2\pi\cdot t}$$

$$\overset{(10b)}{=} \int\limits_{-\infty}^{+\infty} \mathrm{d}f\ C(f)\, e^{+if2\pi\cdot t} = (9a)$$

Many authors introduce the variable substitution

$$\omega = 2\pi f \quad ; \quad \mathrm{d}\omega = 2\pi\,\mathrm{d}f \quad ; \quad D(\omega) := C(\frac{\omega}{2\pi}) \tag{12}$$

in (9). This gives the Fourier transform

$$A(t) = \frac{1}{2\pi} \int\limits_{\omega=-\infty}^{+\infty} \mathrm{d}\omega \; D(\omega)\, e^{+i\omega t} \tag{13a}$$

$$D(\omega) = \int\limits_{t=-\infty}^{+\infty} \mathrm{d}t \; A(t)\, e^{-i\omega t} \tag{13b}$$

with $t \in \mathbb{R}$, $\omega \in \mathbb{R}$, $A(t) \in \mathbb{C}$, $D(\omega) \in \mathbb{C}$.