Elements of Complex Analysis

The most important theorems and tools for applications in physical field theory

Abstract

Some theorems of complex analysis, in particular Cauchy's integral theorem and the residue theorem, are most useful tools for physical field theory. We derive and discuss these mathematical tools without superfluous ballast, but still sufficiently detailed, so that the physicist can not only apply these formulas correctly, but gets as well a working knowledge of their contexts and proofs.

Contents

1. Analytical Functions

Complex numbers may be written as the sum of their real parts x and their imaginary parts iy:

$$z = x + iy \qquad \text{with } x, y \in \mathbb{R},\ z \in \mathbb{C} \tag{1}$$

A complex-valued function $f(z)$ may be written as the sum of it's real part u and it's imaginary part iv:

$$f(x, y) = u(x, y) + iv(x, y) \qquad \text{with } u, v \in \mathbb{R},\ f \in \mathbb{C} \tag{2}$$

We now are looking for a general criterion which answers the question whether f is differentiable at some certain point $x + iy$. Functions g, which are defined for real arguments, are differentiable at a point x, if the left and right limits are finite and identical:

$$\lim_{\varepsilon\to 0} \frac{g(x+\varepsilon) - g(x)}{\varepsilon} = \lim_{\varepsilon\to 0} \frac{g(x) - g(x-\varepsilon)}{\varepsilon} \quad \text{with } \varepsilon > 0 \tag{3}$$

In the plane of complex numbers, the point $z = x + iy$ can be approached from different directions. If the derivative at some point z shall be defined uniquely, then it must not depend on the direction in the complex plane chosen to perform the derivative. As extreme cases, we consider the derivatives parallel to the real axis, and parallel to the imaginary axis. We require that both shall be equal[1]:

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}iy} \tag{4}$$

[1] We apply for the derivative the nomenclature and conventions, which are commonly used in physics. These are different from the conventions commonly used in pure mathematics. For more details, see http://www.astrophys-neunhof.de/mtlg/se77211.pdf

$$\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\mathrm{d}iv}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}iy} + \frac{\mathrm{d}iv}{\mathrm{d}iy} \tag{5}$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} + i\frac{\mathrm{d}v}{\mathrm{d}x} = -i\frac{\mathrm{d}u}{\mathrm{d}y} + \frac{\mathrm{d}v}{\mathrm{d}y} \tag{6}$$

If (6) is split into it's real and imaginary parts, then one gets the Cauchy[2]-Riemann[3] differential equations, which are the basis of the following

Definition: A function $f\colon G \to \mathbb{C}$, which is defined in a region $G \subseteq \mathbb{C}$ of the complex plane, is called **analytic** in the region G, if $f(x,y) = u(x,y) + iv(x,y)$ is a solution of the Cauchy-Riemann differential equations

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}v}{\mathrm{d}y} \tag{7a}$$

$$\text{and} \quad \frac{\mathrm{d}u}{\mathrm{d}y} = -\frac{\mathrm{d}v}{\mathrm{d}x} \tag{7b}$$

for all $x + iy \in G$. Thereby the differential quotients (7a) and (7b) must be finite.

It is a necessary criterion for a function to be uniquely differentiable, that it fulfills the Cauchy-Riemann differential equations. It seems plausible, that this as well is a sufficient criterion, i. e. that the differentials in any direction of the complex plane will be identical, provided they are identical in the directions of the real and the imaginary axes. We spare ourselves the rigorous proof.

The Cauchy-Riemann differential equations (7) impose severe restrictions on differentiable functions. Many seemingly "reasonable" functions do not fulfill (7) and thus are not analytical, for example $f(x,y) = 2x + i3y$ or $f(x,y) = x - iy$. To achieve a better under-

[2] Augustin Louis Cauchy, Aug-21-1789 – May-23-1857
[3] Georg Friedrich Bernhard Riemann, Sep-17-1826 – July-20-1866

standing of (7), we consider f as a two-dimensional vector field, and compute it's divergence and it's rotation. We start with the divergence.

$$f \equiv \begin{pmatrix} u \\ iv \end{pmatrix} \tag{8}$$

$$\begin{aligned} \operatorname{div} f &= \frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\mathrm{d}iv}{\mathrm{d}iy} \\ &= 2 \cdot \frac{\mathrm{d}u}{\mathrm{d}x} = 2 \cdot \frac{\mathrm{d}v}{\mathrm{d}y} \qquad \text{because of (7a)} \end{aligned} \tag{9}$$

Thus both terms of the divergence must be identical. That's an extraordinary requirement, not known from other vector fields. Let's consider the rotation:

$$\begin{aligned} \operatorname{rot} f &= \frac{\mathrm{d}}{\mathrm{d}x} iv - \frac{\mathrm{d}}{\mathrm{d}iy} u \\ i \operatorname{rot} f &= -\frac{\mathrm{d}v}{\mathrm{d}x} - \frac{\mathrm{d}u}{\mathrm{d}y} \\ &= 0 \qquad \text{because of (7b)} \end{aligned} \tag{10}$$

Thus the Cauchy-Riemann differential equation (7b) demands the rotation of the field f to be zero. As an alternative to the extraordinary restriction upon the divergence (9), the condition (7a) could be formulated as the requirement of the vanishing rotation of a field $\tilde{f}$. The field $\tilde{f}$ is defined as f mirrored at the diagonal axis $D\colon a = ia,\ a \in \mathbb{R}$ of the complex plane:

$$\tilde{f} \equiv \begin{pmatrix} iv \\ u \end{pmatrix} \tag{11}$$

$$\begin{aligned} \operatorname{rot} \tilde{f} &= \frac{\mathrm{d}}{\mathrm{d}x} u - \frac{\mathrm{d}}{\mathrm{d}iy} iv \\ &= 0 \qquad \text{because of (7a)} \end{aligned} \tag{12}$$

Obviously the Cauchy-Riemann differential equations impose stronger restrictions upon an analytical function f, than the requirement of zero rotation of that field alone. Stated in other words: There exists a closer connection inbetween the components u and iv of f than inbetween the components of two-dimensional real vector-fields.

Is it worth the effort for a physicist, to consider analytical functions at all, if so many possible functions are excluded by the Cauchy-Riemann differential equations? The answer is a clear yes, because almost all complex functions encountered in physics either are analytic, or have at most a finite number of singular points. To derive maximum advantage from complex analysis, we will need to occupy ourselves with the singularities of complex functions. But prior to that we will state Cauchy's integral theorem.

2. Cauchy's Integral Theorem

Cauchy's integral theorem:
If a function $f(z)\colon G \to \mathbb{C}$ is analytic in all points z of a region $G \subseteq \mathbb{C}$, then the integral along a closed path, which confines that region, is zero:

$$\oint f(z)\mathrm{d}z = 0 \tag{13}$$

This theorem is an immediate consequence of Stokes'[4] theorem. According to Stokes' theorem, the integral (13) of f along a closed path is equal to the surface integral of the rotation of f, computed over the surface enclosed by the path. We stated already in (10), that one of the Cauchy-Riemann differential equations is equivalent

[4] George Gabriel Stokes, Aug-13-1819 – Feb-01-1903

to the requirement that the rotation shall be zero. The rotation of f is zero, wherever f is analytical. Consequently (13) is zero according to Stokes' theorem.

Applying Cauchy's integral theorem, one can read from figure 1:

$$\underset{\text{path 1}}{\int_a^b} f(z)\mathrm{d}z + \underset{\text{path 2}}{\int_b^a} f(z)\mathrm{d}z = \underset{\text{path 1}}{\int_b^a} f(z)\mathrm{d}z + \underset{\text{path 3}}{\int_a^b} f(z)\mathrm{d}z = \underset{\text{path 2}}{\int_b^a} f(z)\mathrm{d}z + \underset{\text{path 3}}{\int_a^b} f(z)\mathrm{d}z = 0 \quad (14)$$

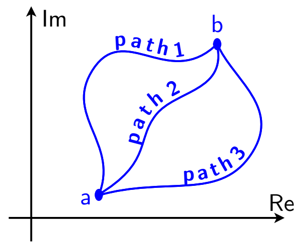

Fig. 1: different integration paths

Two theorems are immediate consequences of (14):

Theorem:
If a function $f(z)\colon G \to \mathbb{C}$ is analytical in all points z of a region $G \subseteq \mathbb{C}$, then the path-integral from a point $a \in G$ to a point $b \in G$ is independent of the integration path, provided the path is completely within G:

$$\underset{\text{path 1}}{\int_a^b} f(z)\mathrm{d}z = \underset{\text{path 2}}{\int_a^b} f(z)\mathrm{d}z \quad (15)$$

Theorem:
If a function $f(z)\colon G \to \mathbb{C}$ is analytic in all points z of a region $G \subseteq \mathbb{C}$, then the path-integral from a point $a \in G$ to a point $b \in G$ is inversely equal to the path-integral from b to a. The paths of the two integrals thereby may be identical or different, provided they nowhere leave G:

$$\int_a^b f(z)\mathrm{d}z = -\int_b^a f(z)\mathrm{d}z \tag{16}$$

3. Singular Points

Cauchy's integral theorem is valid only, if a function is analytic everywhere in a region. But even in case that there are a finite number of singular points within the region, complex analysis has a powerful tool at it's disposal, named residue theorem. First we define the notion "singular point":

Definition: Let a function $f(z) : G \to \mathbb{C}$ be analytic everywhere in a region $G \subseteq \mathbb{C}$ with the exception of a finite number of points a_k. The points $a_k \in G$, at which f is not analytic, are called **singular points** of f. (17)

Three types of singular points can be discerned:

* $f(z) \to +\infty$ or $f(z) \to -\infty$, if z approaches a from an arbitrary direction in the complex plane. This type of singularity is called a "pole".
* $f(z) \to f_0$ with f_0 finite, if z approaches a from an arbitrary direction in the complex plane, but $f(a) \neq f_0$. This is a "removable" singularity. It is removed due to replacement of $f(z)$ at the point a by f_0.

* If z approaches a from different directions of the complex plane, then $f(z)$ converges towards different (finite or infinite) values. This type of singularity is called an "essential singularity".

The type of singularity, which is by far the most important for physical applications, is the pole. There are different orders of poles. The definition of the order of a pole will be given in (21).

4. Laurent Series

The Laurent[5]-series expansion (18) is the essential prerequisite for the residue theorem. We will not delve into it's proof[6].

Theorem: Let a function $f\colon G \to \mathbb{C}$ be analytic everywhere in a region $G \subseteq \mathbb{C}$ of the complex plane with the (possible) exception of a point a. At the point a, the function f may be singular (but this is not necessary). Then $f(z)$ can be expanded in a Laurent series around the point a for all $z \neq a,\ z \in G$:

$$f(z) = \sum_{n=-\infty}^{+\infty} c_n (z-a)^n \quad \text{with } c_n \in \mathbb{C},\ n \in \mathbb{Z} \tag{18}$$

From the Laurent series expansion of a function around the point a, conclusions are possible regarding the type of the singularity of f at this point. The conclusions are stated in the following theorems, which again are cited without proof.

Theorem: If all coefficients c_n with $n < 0$ of the Laurent series (18) are zero, then $f(z)$ is analytic at the point a, or it has there a removable singularity. (19)

[5] Pierre Alphonse Laurent, July-18-1813 – Sep-02-1854

[6] Actually the Laurent series expansion is valid even under less restrictive conditions than stated in the following theorem. But all physical applications meet the restrictive conditions stated in (18).

Theorem: If the coefficient c_m with $m < 0$ of the Laurent series (18) is different from zero, and if all coefficients c_n with $n < m$ are zero, then $f(z)$ has a pole at the point a. (20)

Definition: In this case the pole is called a **pole of order** m. (21)

An example for a pole of order m is $f(z) \equiv 1/(z-a)^m$, $m > 0$, $m \in \mathbb{Z}$.

Theorem: If infinitely many coefficients c_n with $n < 0$ of the Laurent series (18) are different from zero, then $f(z)$ has at the point a an essential singularity. (22)

5. The Residue Theorem

We integrate (18) along a closed path counterclockwise around the point a. The path shall be within the region G, where f is analytic.

$$\oint_{\circlearrowleft} f(z)\mathrm{d}z = \sum_{n=-\infty}^{+\infty} c_n \oint_{\circlearrowleft} (z-a)^n \mathrm{d}z \tag{23}$$

We choose the integration path to be a circle around a with radius $r \in \mathbb{R}$ and angular variable $\varphi \in \mathbb{R}$. Thus the equation and the differential of the integration path become

$$z = a + r \cdot e^{i\varphi} \tag{24a}$$

$$\mathrm{d}z = r \cdot i \cdot e^{i\varphi}\mathrm{d}\varphi\ . \tag{24b}$$

This is inserted into (23):

$$\oint_{\circlearrowleft} f(z)\mathrm{d}z = \sum_{n=-\infty}^{+\infty} c_n \int_{\varphi=0}^{2\pi} (a + re^{i\varphi} - a)^n ri \cdot e^{i\varphi}\mathrm{d}\varphi \tag{25a}$$

$$= \sum_{n=-\infty}^{+\infty} c_n \cdot ir^{n+1} \int_{\varphi=0}^{2\pi} e^{i\varphi(n+1)}\mathrm{d}\varphi \tag{25b}$$

$$= c_{-1} \cdot 2\pi i \tag{25c}$$

The clou of this computation is the integral (25b). Only with $n = -1$ it's value is 2π, for any other n it is zero. Because of $r^{-1+1} = 1$, the surprisingly simple result is (25c). The factor c_{-1}, which is the only one which survives in the integration (25), is appropriately called "residue":

Definition: Let a function $f(z)\colon G \to \mathbb{C}$ be analytic everywhere in a region $G \subseteq \mathbb{C}$ with the only exception of a point a. The integral

$$\mathrm{Res}_f(a) \equiv \frac{1}{2\pi i} \oint_{\circlearrowleft} f(z)\mathrm{d}z\ , \tag{26}$$

which is to be computed along an arbitrary closed path within the region G counterclockwise around the singular point a, is called **residue of f at the point a**.

In (25) we computed the residue along a circular integration path. But definition (26) says that the path is "arbitrary". This can be explained by considering figure 2. In figure 2a, the circular integration path with the singular point a in it's center is sketched. In addition, an arbitrary integration path around

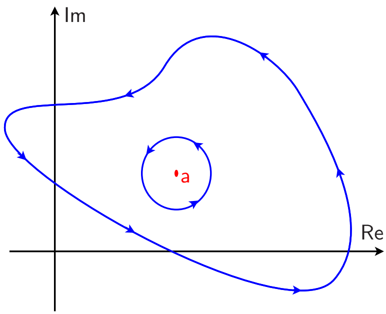

Fig. 2a : Two different integration paths

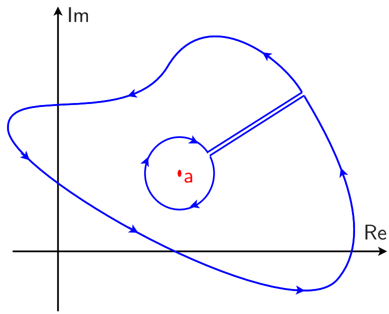

Fig. 2b : The difference of the integration paths

a is indicated. In figure 2b, the difference

$$\underset{\substack{\circlearrowleft \\ \text{fig. 2b}}}{\oint} f(z)\mathrm{d}z = \underset{\substack{\circlearrowleft \\ \text{arb. path,} \\ \text{fig. 2a}}}{\oint} f(z)\mathrm{d}z - \underset{\substack{\circlearrowleft \\ \text{circle,} \\ \text{fig. 2a}}}{\oint} f(z)\mathrm{d}z \tag{27}$$

between the “arbitrary” outer path, and the circular path is shown. The connections between the inner and outer paths are sketched as two closely parallel paths for clarity, but actually these connection shall be along exactly the same path, such that the contributions of the ways in and out shall exactly compensate. The integral fig. 2b is zero according to Cauchy’s integral theorem, as it does not enclose a singular point. Consequently the value of the residue is independent of the integration path around the singular point a, and the wording “arbitrary path” in definition (26) is reasonable.

If f is not singular at a, then the residue is of course zero (as stated by Cauchy’s integral theorem). But it is most useful, that path integrals can be computed by means of (26) even if they enclose a finite number of singular points. This is stated by the

Residue Theorem: Let a function $f\colon G \to \mathbb{C}$ be analytic everywhere in a region $G \subseteq \mathbb{C}$ of the complex plane for all $z \in G$ with the exception of a finite number of points $a_k \in G$. The integral along a closed counterclockwise path in G, which encloses the k singular points a_k, is

$$\oint_{\circlearrowleft} f(z)\mathrm{d}z = 2\pi i \cdot \sum_k \mathrm{Res}_f(a_k) \ . \tag{28}$$

The proof is almost obvious with regard to the previous statements. For $k = 1$ the theorem is trivial, as it then simply reduces to the definition (26) of the residue. If the closed integration path encloses several singular points, then for each singular point a circular path, which encloses only this one singularity, can be computed. The difference between the total integral of (28) and the k circular single integrals is zero, according to figure 2.

6. How to compute Residues

We want to compute integrals of the type $\oint f(z)\mathrm{d}z$. Thus far we only rephrased the problem, but we have not yet solved it. The residue theorem says, that the path integral may be replaced by a sum over residues. But how can a residue be computed? In case that the singularities are poles — but not essential singularities — this is surprisingly simple, by means of the Laurent expansion. We first state a formula for the computation of residues, and will proof it subsequently.

Theorem: Let a function $f\colon G \to \mathbb{C}$, which is defined in a region $G \subseteq \mathbb{C}$, have a pole of order m at the point $a \in G$. The residue of f at the point a can be computed by the following formula:

$$\mathrm{Res}_f(a) = \frac{1}{(m-1)!}\,\frac{\mathrm{d}^{\,m-1}}{\mathrm{d}z^{m-1}}\Big(f(z)\cdot(z-a)^m\Big)\Big|_{z=a} \tag{29}$$

This formula makes the substantial simplification obvious, which the residue theorem brings about. Instead of the need to compute a path integral, which's solution may be a very tough problem, according to (29) one only needs to compute some derivatives; this can always be achieved with little efforts. To prove the formula, we insert for $f(z)$ the Laurent expansion (18):

$$\begin{aligned}\mathrm{Res}_f(a) &= \frac{1}{(m-1)!}\,\frac{\mathrm{d}^{\,m-1}}{\mathrm{d}z^{m-1}}\Big(\sum_{n=-\infty}^{+\infty} c_n(z-a)^n\cdot(z-a)^m\Big)\Big|_{z=a}\\ &= \frac{1}{(m-1)!}\sum_{n=-\infty}^{+\infty} c_n\,\frac{\mathrm{d}^{\,m-1}}{\mathrm{d}z^{m-1}}(z-a)^{m+n}\Big|_{z=a}\end{aligned} \tag{30}$$

All terms, in which the exponent $m+n$ is smaller than the multiplicity $m-1$ of the derivative, become zero in the derivative. Only terms with

$$\begin{aligned}m-1 &\leq m+n\\ -1 &\leq n\end{aligned} \tag{31}$$

do not vanish in the derivative. After the $(m-1)$th derivation, the functional value at $z=a$ shall be taken, as stipulated by the mark $|_{z=a}$. Thereby all terms disappear, which then still contain a factor $(z-a)^p$ with $p>0$. Only terms with

$$\begin{aligned}m-1 &\geq m+n\\ -1 &\geq n\end{aligned} \tag{32}$$

are different from zero after the derivatives have been taken. In total, only the term with $n = -1$ survives in (30), according to the criteria (31) and (32):

$$\begin{aligned} \mathrm{Res}_f(a) &= \frac{1}{(m-1)!}\, c_{-1}\, \frac{\mathrm{d}^{\,m-1}}{\mathrm{d}z^{m-1}} (z-a)^{m-1}\Big|_{z=a} \\ &= \frac{1}{(m-1)!}\, c_{-1}\, (m-1)! \cdot 1 \\ &= c_{-1} \end{aligned} \tag{33}$$

According to (25) and (26), this accords with the definition of the residue. Thus theorem (29) is proved.

Note: In this prove we assumed, that m is finite. Therefore (29) holds — as stated explicitly in the theorem's premises — only for poles, but not for essential singularities.

7. An Example

In physical field theory, often integrals of the form

$$G(x-y) = \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}^4 k}{(2\pi)^4} \, \frac{i \, \exp\{-ik(x-y)\}}{\hbar c \Big(k^0 + \dfrac{\omega_{\boldsymbol{k}}}{c}\Big)\Big(k^0 - \dfrac{\omega_{\boldsymbol{k}}}{c}\Big)} \tag{34}$$

$$\text{with} \quad \frac{\omega_{\boldsymbol{k}}}{c} \equiv +\sqrt{\boldsymbol{k}^2 + M^2 \frac{c^2}{\hbar^2}} \quad ; \quad k \equiv (k^0, k^1, k^2, k^3) \equiv (k^0, \boldsymbol{k})$$

are encountered. (This example is the Greens-function of a Klein-Gordan field with mass M.) At $k^0 = \mp\omega_{\boldsymbol{k}}/c$, the integrand has two poles of first order. Therefore we add a small imaginary term $-i\epsilon$ to $\omega_{\boldsymbol{k}}/c$:

$$\omega_{\boldsymbol{k}}/c \;\to\; \omega_{\boldsymbol{k}}/c - i\epsilon \quad \text{with} \begin{cases} \epsilon \in \mathbb{R} \;,\; \epsilon > 0 \\ \dfrac{\epsilon}{\omega_{\boldsymbol{k}}/c} \le \dfrac{\epsilon}{Mc/\hbar} \ll 1 \end{cases} \tag{35}$$

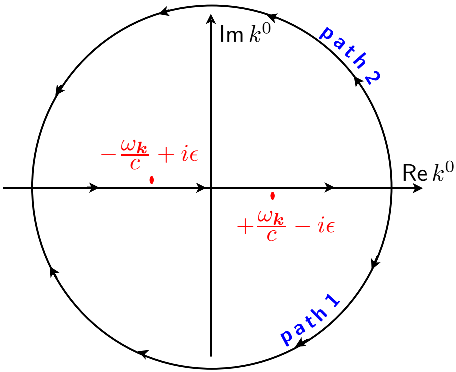

Fig. 3: The two integration paths

Thus the poles, which are indicated in fig. 3 as red points, are shifted from the real axis into the complex plane. Whether, and under which conditions, this measure can be justified, is a physical question. In this circular we occupy ourselves exclusively with the mathematical aspect, i. e. with the computation of the modified integral.

The integral over k^0 can be computed by means of the residue theorem (28). For this purpose, it must be completed to an integral along a closed path. In case $x^0 > y^0$, the integration path can be closed — without change of the integral's value — in the lower complex half-plane (path 1 in fig. 3), because for large real part of k^0 the term $(k^0)^2$ in the denominator, and for large negative imaginary part of k^0 the exponential function in the numerator, will bring about that the integral over the lower semicircle is zero.

Path 1 encloses the pole at $k^0 = +\omega_{\boldsymbol{k}}/c - i\epsilon$. Using the definition

$$f(k^0) \equiv \frac{\exp\{-ik^0(x^0 - y^0)\}}{\hbar c\Big(k^0 + \dfrac{\omega_{\boldsymbol{k}}}{c} - i\epsilon\Big)\Big(k^0 - \dfrac{\omega_{\boldsymbol{k}}}{c} + i\epsilon\Big)} \ , \tag{36}$$

one finds the Greensfunction

$$G(x-y) \overset{x^0 \geq y^0}{=} \frac{i}{(2\pi)^4} \int\limits_{-\infty}^{+\infty} \mathrm{d}^3k \, \exp\{+i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{y})\} \oint\limits_{\circlearrowright} \mathrm{d}k^0 \, f(k^0)$$

$$\overset{(28)}{=} \frac{i}{(2\pi)^4} \int\limits_{-\infty}^{+\infty} \mathrm{d}^3k \, \exp\{+i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{y})\} \Big(-2\pi i \cdot \mathrm{Res}_f(\omega_{\boldsymbol{k}}/c - i\epsilon) \Big) \, .$$

There was a change of sign, because the integration path is followed clockwise, while theorem (28) assumes a counterclockwise integration path. The residue is computed by means of theorem (29). In this example, the order of the pole is $m = 1$:

$$\mathrm{Res}_f(\omega_{\boldsymbol{k}}/c - i\epsilon) \overset{(29)}{=} f(k^0) \cdot (k^0 - \omega_{\boldsymbol{k}}/c + i\epsilon)\Big|_{k^0 = \omega_{\boldsymbol{k}}/c - i\epsilon}$$

$$\overset{(36)}{=} \frac{\exp\{-ik^0(x^0-y^0)\}}{\hbar c \Big(k^0 + \dfrac{\omega_{\boldsymbol{k}}}{c} - i\epsilon \Big)}\Bigg|_{k^0 = \omega_{\boldsymbol{k}}/c - i\epsilon}$$

This results into

$$G(x-y) = \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}^3k}{(2\pi)^3} \, \frac{\exp\{-i(\omega_{\boldsymbol{k}}/c - i\epsilon)(x^0-y^0) + i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{y})\}}{\hbar c(2\omega_{\boldsymbol{k}}/c - 2i\epsilon)} \, .$$

Now ϵ can be neglected versus $\omega_{\boldsymbol{k}}/c$, and the Greensfunction

$$G(x-y) \overset{x^0 \geq y^0}{=} \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}^3k}{(2\pi)^3} \, \frac{\exp\{-i\omega_{\boldsymbol{k}}(x^0-y^0)/c + i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{y})\}}{2\hbar\omega_{\boldsymbol{k}}} \tag{37a}$$

is found. In case $x^0 < y^0$, the integral is closed in the upper complex half-plane (path 2 in Fig. 3). Now the pole at $k^0 = -\omega_{\boldsymbol{k}}/c + i\epsilon$ is enclosed by the integration path:

$$G(x-y) \overset{y^0 \geq x^0}{=} \frac{i}{(2\pi)^4} \int\limits_{-\infty}^{+\infty} \mathrm{d}^3k \, \exp\{+i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{y})\} \oint\limits_{\circlearrowright} \mathrm{d}k^0 \, f(k^0)$$

$$\overset{(28)}{=} \frac{i}{(2\pi)^4} \int\limits_{-\infty}^{+\infty} \mathrm{d}^3k \, \exp\{+i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{y})\} \, 2\pi i \cdot \mathrm{Res}_f(-\omega_{\boldsymbol{k}}/c + i\epsilon)$$

Using

$$\mathrm{Res}_f(-\omega_{\boldsymbol{k}}/c + i\epsilon) \overset{(29)}{=} f(k^0) \cdot (k^0 + \omega_{\boldsymbol{k}}/c - i\epsilon)\Big|_{k^0 = -\omega_{\boldsymbol{k}}/c + i\epsilon}$$

$$\overset{(36)}{=} \frac{\exp\{-ik^0(x^0-y^0)\}}{\hbar c\Big(k^0 - \dfrac{\omega_{\boldsymbol{k}}}{c} + i\epsilon\Big)}\Bigg|_{k^0 = -\omega_{\boldsymbol{k}}/c + i\epsilon}$$

one finds

$$G(x-y) \overset{y^0 \geq x^0}{=}$$

$$= -\int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}^3k}{(2\pi)^3} \, \frac{\exp\{-i(-\omega_{\boldsymbol{k}}/c + i\epsilon)(x^0-y^0) + i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{y})\}}{\hbar c(-2\omega_{\boldsymbol{k}}/c + 2i\epsilon)} \; .$$

As the integration is running symmetrically over all positive and negative wavenumbers $\boldsymbol{k}$, and because of $\omega_{-\boldsymbol{k}} = \omega_{\boldsymbol{k}}$, $\boldsymbol{k}$ and $-\boldsymbol{k}$ may be exchanged. Skipping ϵ, one arrives at

$$G(x-y) \overset{y^0 \geq x^0}{=} \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}^3k}{(2\pi)^3} \, \frac{\exp\{-i\omega_{\boldsymbol{k}}(y^0-x^0)/c + i\boldsymbol{k}(\boldsymbol{y}-\boldsymbol{x})\}}{2\hbar\omega_{\boldsymbol{k}}} \; . \tag{37b}$$

For $x^0 = y^0$, no Greensfunction is defined.